On the structure of t-representable sumsets

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Determining hA

Problem

Given a finite set $A \subseteq \mathbb{Z}$, and an integer $h \geq 1$, determine

$$hA = \underbrace{A + \ldots + A}_{h \text{ times}} = \{c_1 + \ldots + c_h \mid c_1, \ldots, c_h \in A\}.$$

Normalization:

- $A' := A - a_0 = \{a - a_0 \mid a \in A\}$, where $a_0 = \min A$.
- $A'' := \{\frac{a'}{d} \mid a' \in A\}$, where $d = \gcd(A)$.

So suppose wlog: $\{0, m\} \subseteq A \subseteq [m] := \{0, 1, \ldots, m\}$, and $\gcd(A) = 1$.

$$m := \max_{a \in A} a$$

In this case: $A \subseteq 2A \subseteq 3A \subseteq \ldots$ and $hA \subseteq [hm]$.

Representables and exceptions

- Which elements of $[hm]$ are *not* in hA?

Definition

$$\mathcal{P}(A) := \left\{ \sum_{a \in A} k_a\, a \;\middle|\; k_a \in \mathbb{Z}_{\geq 0} \right\} = \bigcup_{h \geq 1} hA$$

$$\mathcal{E}(A) := \mathbb{Z}_{\geq 0} \setminus \mathcal{P}(A) \qquad \text{(exceptional set)}$$

i.e., $n \in \mathcal{E}(A)$ cannot be written as a sum of elements of A

- $hA \subseteq [hm] \setminus \mathcal{E}(A)$.

Exceptional elements

Example 1: $A = \{0, 3, 5\}$

- $\mathcal{P}(A) = \{3x + 5y \mid x, y \in \mathbb{Z}_{\geq 0}\}$
- $8, 9, 10 \in \mathcal{P}(A) \quad \overset{3 \in A}{\Longrightarrow} \quad [8, \infty) \subseteq \mathcal{P}(A)$
- $\mathcal{E}(A) = \{1, 2, 4, 7\}$ (not in $2A$, $3A$, $\ldots$)

In general:

- $\mathcal{E}(A)$ is finite.
- $\mathrm{Fr}(A) := \max\{n \in \mathcal{E}(A)\}$ is the **Frobenius number** of A.

Further exceptions

Example: $A = \{0, 1, 2, 3, 5\}$

- $\mathcal{E}(A) = \varnothing$
- But we know of an element which is never in hA: $2A = [10] \setminus \{9\}$, $3A = [15] \setminus \{14\}$, ...
- $5 - A = \{0, 2, 3, 4, 5\}$, $\quad \mathcal{E}(5 - A) = \{1\}$

Let $m - A = \{m - a \mid a \in A\}$.

Since $hA = hm - (h(m - A))$, if $n \in \mathcal{E}(m - A)$ then $hm - n \notin hA$.

- $hA \subseteq [hm] \setminus \big(\mathcal{E}(A) \cup (hm - \mathcal{E}(m - A))\big)$.

Structure

If:

$$\boxed{hA = [hm] \setminus \big(\mathcal{E}(A) \cup (hm - \mathcal{E}(m-A))\big)} \qquad (*)$$

That'd be nice:

In principle, we could

- Precompute $\mathcal{E}(A)$, $\mathcal{E}(m-A)$;
- Determine hA in linear time $O(hm)$.

Theorem (Nathanson, 1972)

Let $A \subseteq \mathbb{Z}_{\geq 0}$ be a finite set. Then hA satisfies $(*)$ for every

$$h \geq m^2(|A|-1).$$

When hA satisfies $(*)$, hA is said to be **structured**.

"From some point forward, the hA acquire structure, and it persists."

From which point forward?

Problem

Given $A \subseteq \mathbb{Z}_{\geq 0}$, find the smallest $h_1 = h_1(A) \in \mathbb{Z}_{\geq 1}$ s.t. hA is structured $\forall h \geq h_1$.

- $A = \{0 = a_0 < a_1 < \ldots < a_\ell < a_{\ell+1} = m\}, \quad \ell = |A| - 2.$

	Upper bound for structure
Chen–Chen–Wu (2011)	$h_1(A) \leq \sum_{i=2}^{\ell+1}(a_i - 1) - 1$
Granville–Shakan (2020)	$h_1(A) \leq 2\lfloor m/2 \rfloor$
Granville–Walker (2021)	$h_1(A) \leq m - \ell$
Lev (2022)	$h_1(A) \leq \max\{m - \frac{3}{2}(\ell - 1), \frac{2}{3}(m - \ell)\}$, unless A or $m - A = \{0, 1, m - \ell + 1, \ldots, m\}$

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t-representables

$$A = \{0 = a_0 < a_1 < \ldots < a_\ell < a_{\ell+1} = m\}, \quad \ell = |A| - 2$$

hA = # of numbers in $[hm]$ that can be represented at least once as a sum of h elements of A.

$(hA)^{(2)} :=$ # ——— $[hm]$ ——————————— twice ——— — h elements of A.

$(hA)^{(3)} :=$ # ——— $[hm]$ ——————————— thrice ——— — h elements of A. ...

Representation function (of A) **:** $(a_0 = 0)$

$$\rho_{A,h}(n) := \#\{(k_0, \ldots, k_{\ell+1}) \in \mathbb{Z}_{\geq 0}^{\ell+2} \mid k_0 a_0 + \ldots + k_{\ell+1} a_{\ell+1} = n,\ \textstyle\sum_{i=0}^{\ell+1} k_i = h\}$$
$$= \#\{(k_1, \ldots, k_{\ell+1}) \in \mathbb{Z}_{\geq 0}^{\ell+1} \mid k_1 a_1 + \ldots + k_{\ell+1} a_{\ell+1} = n,\ \textstyle\sum_{i=1}^{\ell+1} k_i \leq h\}$$

***t*-representables:** $\boxed{(hA)^{(t)} = \{n \in [hm] \mid \rho_{A,h}(n) \geq t\}}$

Do these sets acquire "structure"?...

t-structure

...Yes!

$$\mathcal{P}_t(A) := \bigcup_{h\geq 1} (hA)^{(t)} \qquad \text{(eventually } t\text{-representables)}$$

$$\mathcal{E}_t(A) := \mathbb{Z}_{\geq 0} \setminus \mathcal{P}_t(A) \qquad \text{(never } t\text{-representables)}$$

Theorem (Nathanson, 2020)

Let $A \subseteq \mathbb{Z}_{\geq 0}$ be a finite set, $t \geq 1$ an integer. Then, $\exists h_t = h_t(A) \in \mathbb{Z}_{\geq 1}$ such that

$$(hA)^{(t)} = [hm] \setminus \big(\mathcal{E}_t(A) \cup (hm - \mathcal{E}_t(m - A))\big)$$

for every $h \geq h_t$. Moreover, $h_t(A) \leq m\ell(tm - 1) + 1$.

- Yang–Zhou (2021): $\quad h_t(A) \leq \sum_{i=2}^{\ell+1} (ta_i - 1) - 1$

Statement of the main theorem

- $A = \{0 = a_0 < a_1 < \ldots < a_\ell < a_{\ell+1} = m\}, \quad \ell = |A| - 2$
- $(hA)^{(t)}$ is structured for every $h \geq h_t(A)$.

Main Theorem

We have

$$\boxed{h_t(A) \leq C_{A,t} \cdot \frac{1}{e} m\ell t^{1/\ell}}$$

where

$$C_{A,t} \leq \left(1 + \frac{4}{\ell}\right)\frac{e}{t^{1/\ell}} + \left(1 + \frac{2}{\ell}\right)\frac{1 + (\log 4\ell)/\ell}{\min\{a_1, m - a_\ell\}}.$$

- $C_{A,t} \leq 3e$ for $\ell \geq 4$;
- $C_{A,t} \lesssim 1$ as $\ell \to \infty$, $t^{1/\ell} \to \infty$.

Two remarks

$$\boxed{h_t(A) \le C_{A,t} \cdot \frac{1}{e} m\ell t^{1/\ell}}$$

$C_{A,t} \le 3e$ for $\ell \ge 4$;
$C_{A,t} \lesssim 1$ as $\ell, t^{1/\ell} \to \infty$.

- Comparison: For $\ell \ge 4$, we have $h_t(A) \le 3m\ell t^{1/\ell}$.

$$\overbrace{\sum_{i=2}^{\ell+1} (ta_i - 1) - 1}^{\text{Yang–Zhou (2021)}} \; \ge \; mt + \left(\frac{\ell^2 - 3}{2}\right) t.$$

We check that $3m\ell t^{1/\ell} \le mt + (\frac{\ell^2-3}{2})t$ for $t \ge 4\ell$.

- Asymptotic sharpness: For $A = \{0, 1, m - \ell + 1, \ldots, m\}$, with $\ell = \ell(m) := \lfloor m^{1/2.01} \rfloor$, and $t = t(m) := \binom{\ell + \lfloor m^{1/2} \rfloor}{\ell}$, we have

$$h_t(A) \ge (1 - o_{m\to\infty}(1)) \, \frac{1}{e} m\ell t^{1/\ell}.$$

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The t**-Frobenius number** is defined as $\mathrm{Fr}_t(A) := \max\{n \in \mathcal{E}_t(A)\}$.

$\mathrm{Fr}_t(A)$ *is well-defined* (i.e., $|\mathcal{E}_t(A)| < \infty$).

- Let $\rho_A(n) = \#\{(k_1, \ldots, k_{\ell+1}) \in \mathbb{Z}_{\geq 0}^{\ell+1} \mid k_1 a_1 + \ldots + k_{\ell+1} a_{\ell+1} = n\}$ or

$$\rho_A(n) := \lim_{h \to \infty} \rho_{A,h}(n).$$

So $\mathcal{E}_t(A) = \{n \in \mathbb{Z}_{\geq 0} \mid \rho_A(n) < t\}$.

Estimating $\mathrm{Fr}_t(A)$ effectively (sketch)

By counting lattice points, we can obtain effective bounds:

Lemma 1

Let $A = \{0 = a_0 < a_1 < \ldots < a_\ell < a_{\ell+1} = m\}$. *If* $a_1 = 1$, *then*

$$\rho_A(n) \ \geq \ \frac{1}{\ell!} \frac{n^\ell}{a_1 \cdots a_{\ell+1}}.$$

(Idea) Let

$$\Delta := \{(x_2, \ldots, x_{\ell+1}) \in (\mathbb{R}_{\geq 0})^\ell \mid x_2 a_2 + \ldots + x_{\ell+1} a_{\ell+1} \leq n\}.$$

If $a_1 = 1$, then $\rho_A(n) = |\Delta \cap \mathbb{Z}^\ell| \geq \mathrm{vol}_{\mathbb{R}^\ell}(\Delta)$. …

Corollary 2

$$\mathrm{Fr}_t(A) \leq (a_1 \cdots a_{\ell+1})^{1/\ell} (\ell!)^{1/\ell} (t-1)^{1/\ell} + (a_1 - 1) \sum_{j=2}^{\ell+1} a_j.$$

The t-structure theorem

Let $h_t(A)$ be the minimum number such that

$$(hA)^{(t)} = [hm] \setminus \big(\mathcal{E}_t(A) \cup (hm - \mathcal{E}_t(m-A))\big)$$

for every $h \geq h_t(A)$.

Main Lemma (t-Frobenius $< \infty \implies t$-structure)

$$h_t(A) \leq \left\lfloor \frac{\mathrm{Fr}_t(A) + m}{a_1} \right\rfloor + \left\lfloor \frac{\mathrm{Fr}_t(m-A) + m}{m - a_\ell} \right\rfloor .$$

(Pf) <u>of Main Theorem:</u> Corollary 2 + calculations yield

$$\frac{\mathrm{Fr}_t(A) + m}{a_1} + \frac{\mathrm{Fr}_t(m-A) + m}{m - a_\ell} \leq C_{A,t}\, \frac{1}{e} m\ell t^{1/\ell}. \quad \square$$

Main idea in the proof

Let $A = \{0 = a_0 < a_1 < \ldots < a_\ell < a_{\ell+1} = m\}$.

Take $n \in \mathbb{Z}_{\geq 0}$. A representation of n by A is given by

$$n = k_1 a_1 + \ldots + k_{\ell+1} a_{\ell+1} \qquad (k_i \geq 0)$$

$$\sum k_i = h$$

- Which representation maximizes $\sum_{i=1}^{\ell+1} k_i$?
  $\longrightarrow \quad k_1 = n/a_1, \quad k_2 = \ldots = k_{\ell+1} = 0$
- So for any representation of n, we have $\sum k_i \leq n/a_1$
- Hence, $\boxed{n \in \mathcal{P}_t(A) \implies n \in (hA)^{(t)} \text{ for every } h \geq n/a_1}$.

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From $\mathbb{Z}$ to $\mathbb{Z}^d$

	$A \subseteq \mathbb{Z}$	$A \subseteq \mathbb{Z}^d$
Ambient	$[m]$	convex hull $\mathcal{H}(A)$
	$\gcd(A)$	lattice $\Lambda_A := \mathrm{span}_{\mathbb{Z}}(A)$
	$\mathbb{Z}_{\geq 0}$	lattice cone $\Lambda_A \cap \mathcal{C}_A$, w/ $\mathcal{C}_A := \{\sum_{\underline{a} \in A} c_{\underline{a}} \underline{a} \mid c_{\underline{a}} \in \mathbb{R}_{\geq 0}\}$
Extremes	$\{0, m\}$	"corners" $\underline{0} \in \mathrm{ex}(\mathcal{H}(A))$

- t-structure in $\mathbb{Z}^d$: Defining $\mathcal{E}_t(A) := (\Lambda_A \cap \mathcal{C}_A) \setminus \bigcup_{h \geq 1} (hA)^{(t)}$,

$$(hA)^{(t)} = (h\mathcal{H}(A) \cap \Lambda_A) \setminus \left(\mathcal{E}_t(A) \cup \bigcup_{\underline{v} \in \mathrm{ex}(\mathcal{H}(A))} (h\underline{v} - \mathcal{E}_t(\underline{v} - A)) \right)$$

Structure

Given $A \subseteq \mathbb{Z}^d$ with $\underline{0} \in \mathrm{ex}(\mathcal{H}(A))$, let $h_t = h_t(A)$ be the smallest integer such that:

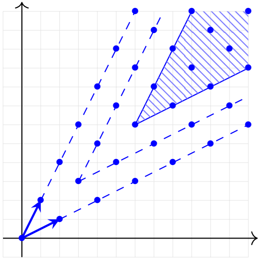
$$(hA)^{(t)} = (h\mathcal{H}(A) \cap \Lambda_A) \setminus \left(\mathcal{E}_t(A) \cup \bigcup_{\underline{v} \in \mathrm{ex}(\mathcal{H}(A))} (h\underline{v} - \mathcal{E}_t(\underline{v} - A)) \right)$$

- Granville–Shakan (2020): $h_1(A) < \infty$.
- Granville–Shakan–Walker (2021):

$$h_1(A) \leq (d+1)2^{11d^2} d^{12d^6} |A|^{3d^2} \left(\max_{\underline{a},\underline{b} \in A} \|\underline{a} - \underline{b}\|_\infty \right)^{8d^6}$$

- T. (2023+): $\forall t \geq 1$, $h_t(A) < \infty$.

Main ideas in the proof



- Lattice cone \ (P + Lattice cone) = finite union of lower dimension cones

- Show that $\exists N = N_{A,t} \in \mathbb{N}$ such that

$$(N\mathcal{H}(A) \cap \Lambda_A) \setminus \mathcal{E}_t(A) + A = ((N+1)\mathcal{H}(A) \cap \Lambda_A) \setminus \mathcal{E}_t(A)$$

(Pf) essentially, induction on d.

Rem.: $N_{A,t} \approx \frac{\mathrm{Fr}_t(A)+m}{m}$

- Apply this to $\underline{v} - A$ for each $\underline{v} \in \mathrm{ex}(\mathcal{H}(A))$, then "glue" everything together. $\square$

Thank you!