From ABC to L: On singular moduli and Siegel zeros

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IUT Summit 2021

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2. Lower bounds for $h(D)$
3. The Siegel zero connection
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"No Siegel zeros" for $D < 0$

$$\iff$$

Lower bounds for "class number $\times$ ~~regulator~~" of $\mathbb{Q}(\sqrt{D})$

Notation

- $D < 0$ fundamental discriminant (so $\sqrt{D} \notin \mathbb{R}$, $D = \operatorname{disc} \mathbb{Q}(\sqrt{D})$);
- $\mathrm{C}\ell(D)$ = class group of $\mathbb{Q}(\sqrt{D})$;
- $h(D) = |\mathrm{C}\ell(D)|$ class number of $\mathbb{Q}(\sqrt{D})$;
- j = j-invariant function:

$$j(\tau) := \frac{\left(1 + 240 \sum_{n\geq 1} \left(\sum_{d|n} d^3\right) q^n\right)^3}{q \prod_{n\geq 1}(1-q^n)^{24}} \qquad \left(\Im(\tau) > 0,\ q = e^{2\pi i\tau}\right)$$

$$\boxed{\begin{aligned} j(\tau+1) &= j(\tau), \\ j(-1/\tau) &= j(\tau); \end{aligned}}$$

$$\boxed{\begin{gathered} j(i) = 1728, \\ j(e^{2\pi i/3}) = 0, \\ \text{“}j(i\infty) = 1\infty\text{”}; \end{gathered}}$$

<u>q-expansion of j</u>:

$$j(\tau) = \frac{1}{q} + 744 + 196884q + \cdots$$

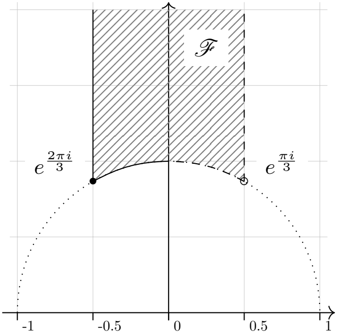
Singular moduli (1/2)

Let $\tau \in \mathfrak{h}$ $(\Im(\tau) > 0)$

- CM-point: $\tau \mid A\tau^2 + B\tau + C = 0$ $\left(\begin{matrix} A, B, C \in \mathbb{Z},\ A > 0, \\ \gcd(A, B, C) = 1,\ \text{unique} \end{matrix}\right)$
- Singular modulus: $j(\tau)$ $(j = j\text{-invariant}, \tau \text{ a CM-point})$

CM-points	$\longleftrightarrow$	Quadratic forms
$\tau \in \mathfrak{h}$		$Ax^2 + Bxy + Cy^2$

- $\operatorname{disc}(\tau) := B^2 - 4AC$
- $\tau \sim \tau' \iff (A, B, C) \sim (A', B', C')$
- $\tau \in \mathscr{F} \iff (A, B, C)$ *reduced*



Singular moduli (2/2)

$$\underline{\text{Heegner points } \Lambda_D} \qquad \underline{\text{Reduced prim. quad. forms of disc. } D}$$

$$\left\{\begin{matrix}\text{CM-points } \tau \in \mathscr{F}, \\ \text{disc}(\tau) = D\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}(a,b,c) := ax^2 + bxy + cy^2 \\ \text{s.t. } b^2 - 4ac = D, \text{ and} \\ -a < b \leq a < c \text{ or } 0 \leq b \leq a = c\end{matrix}\right\}$$

$$\Cup \qquad\qquad\qquad \Cup$$

$$\boxed{\tau_D := \underbrace{\frac{\sqrt{D}}{2}}_{D\equiv 0(4)} \text{ or } \underbrace{\frac{-1+\sqrt{D}}{2}}_{D\equiv 1(4)}} \leftrightarrow \underbrace{\text{Principal form}}_{(1,0,-\frac{D}{4}) \text{ or } (1,1,\frac{1-D}{4})} \qquad \mathbb{Z}[\tau_D] = \mathcal{O}_{\mathbb{Q}(\sqrt{D})}$$

Write $\mathrm{H}_D :=$ Hilbert class field of $\mathbb{Q}(\sqrt{D})$.

- $\mathrm{H}_D = \mathbb{Q}(\sqrt{D}, j(\tau_D)) \quad \left([\mathrm{H}_D : \mathbb{Q}(\sqrt{D})] = [\mathbb{Q}(j(\tau_D)) : \mathbb{Q}] = h(D)\right)$
- $\{j(\tau) \mid \tau \in \Lambda_D\} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{D}))$-conjugates of $j(\tau_D)$
- $j(\tau_D)$ is an algebraic **integer!**

The equation $x^3 - Dy^2 = 1728$

Consider the Diophantine equation $x^3 - Dy^2 = 1728 \quad (= j(i))$.

- Factorization of differences of singular moduli
- Solutions of the type $(j(\tau_D), j(\tau_D) - 1728) = (x^3, Dy^2)$
- Idea: $ABC \implies$ few such solns $\implies$ **lower bounds for** $h(D)$

ABC : $(a + b = c)$

$a, b, c \in \mathbb{Z}$ coprime. $\forall \varepsilon > 0$, $\exists C_\varepsilon > 0$ s.t. $\max\{|a|, |b|, |c|\} < C_\varepsilon \Big(\prod_{p \mid abc} p \Big)^{1+\varepsilon}$

Example (Class number 1 problem)

Gross–Zagier: $h(D) = 1 \implies x,\ y\sqrt{|D|} \in \mathbb{Z}$. However, in this case,

$$|j(\tau_D)| \leq \max\{|x|^3, |y\sqrt{D}|^2, 1728\} \overset{\text{abc}}{\leq} C_\varepsilon \cdot |x \cdot y\sqrt{D}|^{1+\varepsilon}$$
$$\leq C_\varepsilon \cdot |j(\tau_D)|^{\frac{5}{6}(1+\varepsilon)},$$

so there can only be finitely many $D < 0$ with $h(D) = 1$. △

How does $h(D)$ grow?

What do we know?

- $h(D) \to \infty$ as $D \to -\infty$ (Heilbronn, 1934)
  - List $h(D)$ (small values of $|D|$)
  - Estimate $h(D)$ (large values of $|D|$)

- $-D \in \{3, 4, 7, 8, 11, 19, 43, 67, 163\}$ are the *only* D s.t. $h(D) = 1$
  - Solved by Heegner (∼1952) [based on Weber's work]
  - Reworked and clarified by Stark, Birch (1967∼9)
  - Baker's solution (∼1970) [based on Baker's theorem]

- $h(D)$ grows *roughly* like $|D|^{1/2}$ (GRH $\implies$ err $\approx (\log\log|D|)^{\pm 1}$)
  - Hecke (1917): If $L(s, \chi_D) \neq 0$ **near** 1, then $\dfrac{h(D)}{|D|^{1/2}} \gg (\log|D|)^{-1}$
  - Landau (1918): $\max\left\{\dfrac{h(D_1)}{|D_1|^{1/2}}, \dfrac{h(D_2)}{|D_2|^{1/2}}\right\} \gg (\log|D_1 D_2|)^{-1}$
  - Siegel (1935): $\dfrac{h(D)}{|D|^{1/2}} \gg_\varepsilon |D|^{-\varepsilon}$ (unconditional but ineffective)

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The Granville–Stark Theorem

Theorem (Granville–Stark, 2000)

Uniform abc-conj. $\Longrightarrow$ *"No Siegel zeros" for* $L(s, \chi_D),\ D < 0$

- Study $x^3 - y^2 = 1728$
- Uniform ABC for number fields (U-abc)
- U-abc $\Longrightarrow$ lower bounds for $h(D)$
- Lower bounds $h(D)$ $\Longleftrightarrow$ "no Siegel zeros"

A. Granville

H. Stark

The equation $x^3 - y^2 = 1728$

- Solutions of the type $(j(\tau_D), j(\tau_D) - 1728) = (x^3, y^2)$

$$\underline{\textbf{ABC}} \text{ (for } \mathbb{Q}) : \quad \log\max\{|a|,|b|,|c|\} < (1+\varepsilon) \underbrace{\boxed{\sum_{p|abc} \log p}}_{\text{log-conductor}} + O_\varepsilon(1)$$

- Applying the same logic for $x^3 - y^2 - 1728 = 0$, we would get:

$$\begin{aligned} \log\max\{|x|^3,|y|^2\} &\leq \log\max\{|x|^3,|y|^2,1728\} \\ &\overset{\text{abc}}{\leq} (1+\varepsilon)\log\text{-cond}(12|xy|) + \textbf{error} \\ &\leq \frac{5}{6}(1+\varepsilon)\log\max\{|x|^3,|y|^2\} + \textbf{error} \end{aligned}$$

How to make this precise? $\longrightarrow$ $\boxed{\log\max\{|x|^3,|y|^2\} < (6+\varepsilon')\cdot\textbf{error}}$

ABC for number fields (1/2)

Let $K/\mathbb{Q}$ be a NF, $\mathbb{V}(K)$ its places, $\mathbb{V}(K)^{\text{non}} \subseteq \mathbb{V}(K)$ non-arch. places.

For a point $P = [x_0 : \cdots : x_n] \in \mathbb{P}^n_K$, define:

- (naïve, abs, log) <u>height</u> $\mathrm{ht}(P)$

$$\boxed{\frac{1}{[K:\mathbb{Q}]} \sum_{v \in \mathbb{V}(K)} \log\left(\max_i\{\|x_i\|_v\}\right)}$$

- (log) <u>conductor</u> $\mathcal{N}_K(P)$

$$\boxed{\frac{1}{[K:\mathbb{Q}]} \sum_{\substack{v \in \mathbb{V}(K)^{\text{non}} \\ \exists i,j \leq n \text{ s.t.} \\ v(x_i) \neq v(x_j)}} f_v \log(p_v)}$$

$$\updownarrow$$

$$\begin{cases} v \sim \mathfrak{p} = \mathfrak{p}_v \\ p_v \sim \mathfrak{p}_v \cap \mathbb{Q} \\ f_v := [K_v : \mathbb{Q}_{p_v}] \end{cases}$$

For $a, b, c \in \mathbb{Z}$ coprime,

- $\mathrm{ht}([a:b:c]) = \log\max\{|a|,|b|,|c|\}$
- $\mathcal{N}_{\mathbb{Q}}([a:b:c]) = \log\left(\prod_{p|abc} p\right)$

For $\alpha \in \overline{\mathbb{Q}}$, $\mathrm{ht}(\alpha) := \mathrm{ht}([\alpha:1])$.

$$\boxed{\alpha \text{ integral} \Rightarrow \mathrm{ht}(\alpha) = \frac{1}{|\mathcal{A}|} \sum_{\alpha^* \in \mathcal{A}} \log^+|\alpha^*|}$$

$\mathcal{A} = \{\text{conjugates of } \alpha\}$

ABC for number fields (2/2)

abc for number fields

Fix $K/\mathbb{Q}$ a number field. Then, for every $\varepsilon > 0$, there is $\mathcal{C}(K,\varepsilon) \in \mathbb{R}_+$ such that, $\forall a,b,c \in K \mid a+b+c=0$, we have

$$\mathrm{ht}([a:b:c]) < (1+\varepsilon)\Big(\mathcal{N}_K([a:b:c]) + \log(\mathrm{rd}_K)\Big) + \mathcal{C}(K,\varepsilon),$$

where $\mathrm{rd}_K := |\Delta_K|^{1/[K:\mathbb{Q}]}$ *is the* root-discriminant *of* K.

- Granville–Stark's idea: Solve $x^3 - y^2 = j(i)$ in diff.s of singular moduli:

$$(j(\tau_D),\, j(\tau_D) - 1728) = (x^3, y^2) \qquad \boxed{x^3 - y^2 = 1728}$$

$$j(\tau_D) \in \mathrm{H}_D \qquad \widetilde{\mathrm{H}}_D := \mathrm{H}_D(x,y) \quad ([\widetilde{\mathrm{H}}_D : \mathrm{H}_D] \leq 6)$$

Lemma

$$\mathrm{rd}_{\widetilde{\mathrm{H}}_D} \leq 6\sqrt{|D|}$$

Proposition (for $0 < \varepsilon < \frac{1}{5.01}$)

$$\mathrm{ht}(j(\tau_D)) < \frac{3}{1-5\varepsilon}\big((1+\varepsilon)\log|D| + 2\mathcal{C}(\widetilde{\mathrm{H}}_D,\varepsilon)\big) + O(1)$$

What do we get?

Thus, for every small $\varepsilon > 0$, we have:

$$\limsup_{D\to-\infty} \frac{\mathrm{ht}(j(\tau_D))}{3\log|D|} \leq \frac{1+\varepsilon}{1-5\varepsilon} + \limsup_{D\to-\infty} \frac{2\mathcal{C}(\widetilde{\mathrm{H}}_D,\varepsilon)}{(1-5\varepsilon)\log|D|}.$$

How to make progress?

$$\left(\text{Keep in mind: } \begin{array}{c}[\widetilde{\mathrm{H}}_D:\mathbb{Q}] \asymp h(D), \\ \log(\mathrm{rd}_{\widetilde{\mathrm{H}}_D}) = \log|D| + O(1)\end{array}\right)$$

- Uniformity: $\mathcal{C}(K,\varepsilon) = \mathcal{C}(\varepsilon)$ $\quad (\implies \limsup \leq 1)$
- **Weak uniformity**: $\mathcal{C}(K,\varepsilon) = o_\varepsilon(\log \mathrm{rd}_K)$ $\quad (\implies \limsup \leq 1)$
- O-weak uniformity: $\mathcal{C}(K,\varepsilon) = O_\varepsilon(\log \mathrm{rd}_K)$ $\quad (\implies \limsup < +\infty)$
- [IUTchIV] *ABC*:† $\mathcal{C}(K,\varepsilon) = O_\varepsilon([K:\mathbb{Q}]^{4+\varepsilon})$ (not strong enough!)

† Ignoring "compactly boundedness" condition.

Lower bounds for $h(D)$

Lemma [G–S]

$$\text{(weak) uniform } ABC \implies \mathrm{ht}(j(\tau_D)) \leq (3 + o(1)) \log |D|$$

However,

$$\mathrm{ht}(j(\tau_D)) = \frac{1}{h(D)} \sum_{(a,b,c)} \log^+ |j(\tau_{(a,b,c)})|$$

$$= \frac{1}{h(D)} \sum_{(a,b,c)} \frac{\pi\sqrt{|D|}}{a} + O(1),$$

$$ax^2 + bxy + cy^2$$
reduced,
$$b^2 - 4ac = D$$

$$\tau_{(a,b,c)} = \frac{-b + \sqrt{D}}{2a}$$

and thus:

Theorem [G–S]

$$h(D) \overset{\text{U-}abc}{\geq} \left(\frac{\pi}{3} + o(1)\right) \frac{\sqrt{|D|}}{\log |D|} \sum_{(a,b,c)} \frac{1}{a}$$

q-expansion!

$$\boxed{j(\tau) = \frac{1}{q} + O(1)}$$

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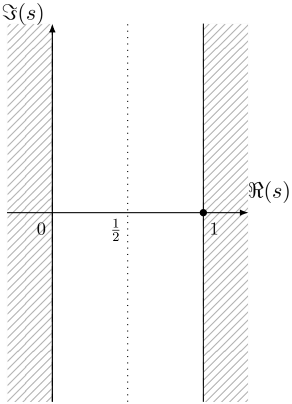
Critical strip

The **Dirichlet L-function** of a primitive character $\chi \pmod q$:

$$L(s,\chi) := \sum_{n\geq 1} \frac{\chi(n)}{n^s}, \qquad (\Re(s) > 1)$$

- Euler product ($\Re(s) > 1$):
  - $L(s,\chi) = \prod_p \frac{1}{1-\chi(p)p^{-s}}$
- Functional equation:
  - $L^*(s,\chi) := \gamma_\chi(s)\, L(s,\chi)$ is entire
  - $L^*(s,\chi) = W(\chi)\, L^*(1-s,\overline{\chi})$

    (where $|W(\chi)| = 1$)
- Dirichlet's PNT:
  - $L(1+it,\chi) \neq 0$, $\forall t \in \mathbb{R}$.



Classical zero-free regions

[Gronwall 1913, Landau 1918, Titchmarsh 1933]

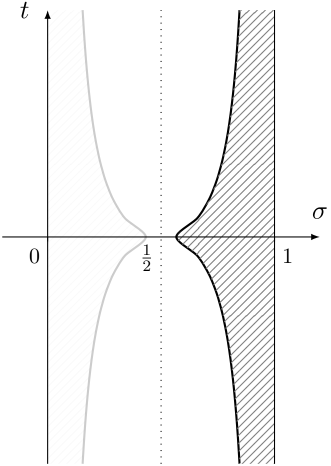
Write $s = \sigma + it$ ($\sigma = \Re(s)$, $t = \Im(s)$), and let $\chi \pmod q$ be a primitive character.

There exists $A > 0$ such that, in the region

$$\left\{ s \in \mathbb{C} \;\middle|\; \sigma \geq 1 - \frac{A}{\log q(|t|+2)} \right\},$$

the function $L(s, \chi)$ has:

- (χ complex) no zeros;
- (χ real) at most one zero, which is necessarily *real* and *simple* – the so-called **Siegel zero**.



Real primitive Dirichlet characters

$$\left\{\begin{matrix}\text{Real primitive}\\ \text{Dirichlet characters}\end{matrix}\right\} \longleftrightarrow \left\{\left(\frac{D}{\cdot}\right),\ \begin{matrix}D \text{ fundamental}\\ \text{discriminant}\end{matrix}\right\}$$

	$\mathbb{Q}(\sqrt{D})$	$\chi_D \pmod{\lvert D\rvert}$
Factorization	$D = D_1 \cdots D_t$	$\chi_D = \chi_{D_1} \cdots \chi_{D_t}$
Ramification	(p) split	$\chi_D(p) = 1$
	(p) inert	$\chi_D(p) = -1$
	(p) ramified	$\chi_D(p) = 0$
∞	Real / Complex	$\chi_D(-1) = 1$ / $\chi_D(-1) = -1$
L-function	$\zeta_{\mathbb{Q}(\sqrt{D})}(s) = \sum \mathrm{N}(\mathfrak{a})^{-s}$	$L(s, \chi_D) = \sum \chi_D(n) n^{-s}$

Quadratic reciprocity $\iff$ $\zeta_{\mathbb{Q}(\sqrt{D})}(s) = \zeta(s) L(s, \chi_D)$

“No Siegel zeros” for $D < 0$

Conjecture (“no Siegel zeros” for $D < 0$)

There is $\delta > 0$ such that $L(\beta, \chi_D) \neq 0$ for $1 - \frac{\delta}{\log |D|} \leq \beta < 1$.

Equivalently:

1. $\frac{L'}{L}(1, \chi_D) \ll \log |D|$
2. $\mathrm{ht}(j(\tau_D)) \ll \log |D|$
3. $h(D) \gg \frac{\sqrt{|D|}}{\log |D|} \sum_{(a,b,c)} \frac{1}{a}$

There are precise relations between $\frac{L'}{L}(1, \chi_D)$, $\mathrm{ht}(j(\tau_D))$, and $h(D)$:

$h(D) \longleftrightarrow \mathrm{ht}(j(\tau_D))$	(seen)
$\boxed{\mathrm{ht}(j(\tau_D)) \longleftrightarrow \frac{L'}{L}(1, \chi_D)}$	(Thm 1)
$\frac{L'}{L}(1, \chi_D) \longleftrightarrow$ Siegel zero	(“tricky”)

① $\iff$ ② (Theorem 1)

Theorem 1 (T., 2020+)

For fundamental discriminants $D < 0$,

$$\frac{L'}{L}(1, \chi_D) = \frac{1}{6}\,\mathrm{ht}(j(\tau_D)) - \frac{1}{2}\log|D| + \underbrace{C + o_{D\to-\infty}(1)}_{\textbf{Error term}}$$

where $C = -1.057770\ldots$.

†**Remark.** (Colmez, 1993) Let $E_D/\mathbb{C}$ be an ell. curve w/ CM by $\mathbb{Z}[\tau_D]$. Then:

$$-2\,\mathrm{ht}_{\mathrm{Fal}}(E_D) = \frac{1}{2}\log|D| + \frac{L'}{L}(0, \chi_D) + \log 2\pi.$$

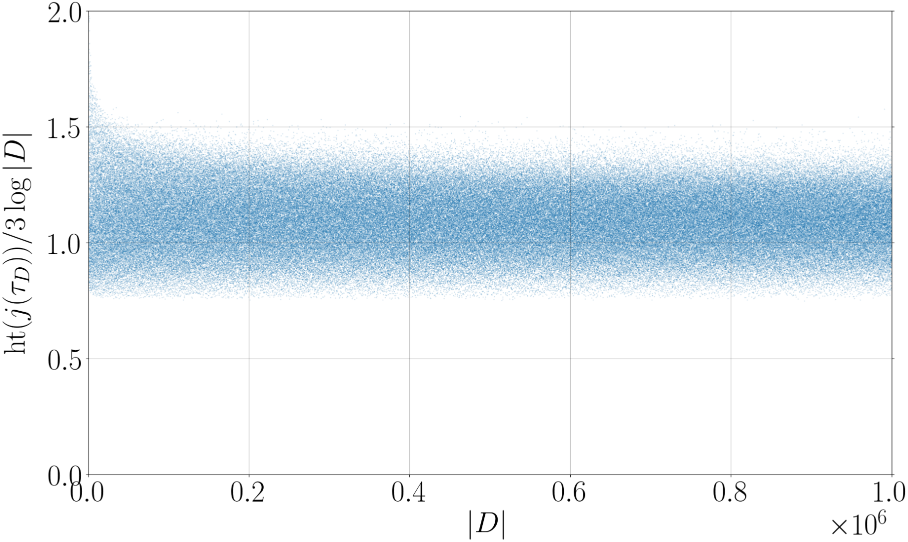
- Colmez + Thm 1 $\implies$ $|\mathrm{ht}_{\mathrm{Fal}}(E_D) - \frac{1}{12}\mathrm{ht}(j(\tau_D))| = 2.65537\ldots + o_{D\to-\infty}(1)$

	Error term
Colmez (1993)	$O(\log \mathrm{ht}(j(\tau_D)))$
G–S (2000)	$O(\log\log|D|)$

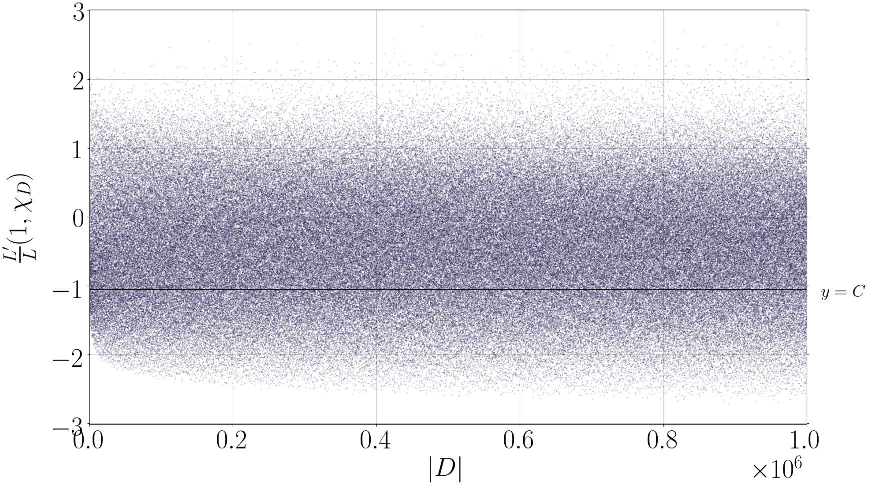
	Error term
Dimitrov (2016)*	$O(1)$
T. (2020+)	$\mathbf{-1.05\ldots + o(1)}$

*Unpublished.

Graph of $\mathrm{ht}(j(\tau_D))$



Graph of $\frac{L'}{L}(1, \chi_D)$



$C = -1.057770\ldots$

Euler–Kronecker constants

The Dedekind ζ-function of $\mathbb{Q}(\sqrt{D})$:

$$\zeta_{\mathbb{Q}(\sqrt{D})}(s) := \sum_{\mathfrak{a}\subseteq\mathcal{O}_{\mathbb{Q}(\sqrt{D})}} \frac{1}{\mathrm{N}(\mathfrak{a})^s} \qquad \left(\begin{aligned} &= \zeta(s)L(s,\chi_D) \\ &= \sum_{\mathscr{A}\in\mathrm{C}\ell(D)} \zeta(s,\mathscr{A}) \end{aligned} \right)$$

where $\zeta(s,\mathscr{A}) = \sum_{\substack{\mathfrak{a}\subseteq\mathscr{A} \\ \mathfrak{a}\text{ integral}}} \frac{1}{\mathrm{N}(\mathfrak{a})^s}$ for $\mathscr{A}\in\mathrm{C}\ell(D)$ — (partial zeta function)

In general, as $s\to 1$:

- $\zeta_K(s) = \frac{c_{-1}}{s-1} + c_0 + O(s-1)$
- $\frac{\zeta_K'}{\zeta_K}(s) = -\frac{1}{s-1} + \gamma_K + O(s-1)$
- $\zeta_K(s,\mathscr{A}) = \frac{\varkappa_K}{s-1} + \varkappa_K\mathfrak{K}(\mathscr{A}) + O(s-1)$

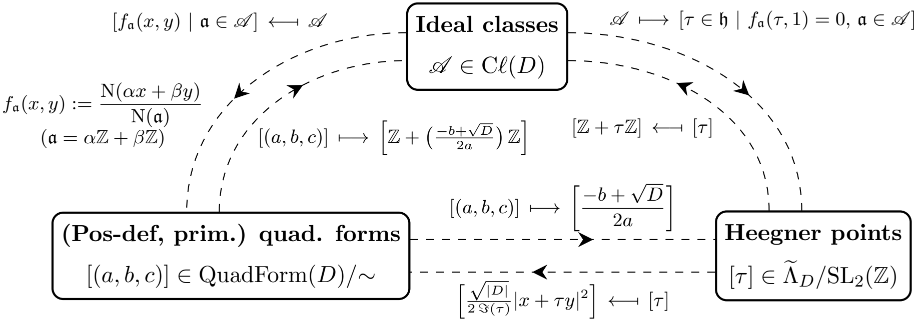
(Ihara, 2006)
Euler–Kronecker:
$\gamma_K := c_0/c_{-1}$

Kronecker limits:
$\mathfrak{K}(\mathscr{A}),\ \mathscr{A}\in\mathrm{C}\ell_K$

$$\gamma_K = \frac{1}{h_K}\sum_{\mathscr{A}\in\mathrm{C}\ell_K}\mathfrak{K}(\mathscr{A})$$

$$\gamma + \frac{L'}{L}(1,\chi_D) = \frac{1}{h(D)}\sum_{\mathscr{A}\in\mathrm{C}\ell(D)}\mathfrak{K}(\mathscr{A})$$

Correspondence for $D < 0$ (Ideals–Forms–Points)



Partial zeta function		Epstein zeta function		real-analytic Eisenstein series
$\zeta(s, \mathscr{A})$	$\leftrightarrow$	$Z_{[(a,b,c)]}(s)$	$\leftrightarrow$	$E([\tau], s)$
$\displaystyle\sum_{\substack{\mathfrak{a} \subseteq \mathscr{A} \\ \mathfrak{a}\text{ integral}}} \frac{1}{\mathrm{N}(\mathfrak{a})^s}$		$\displaystyle\sum_{\substack{(x,y)\in\mathbb{Z}^2 \\ (x,y)\neq(0,0)}} \frac{1}{(ax^2+bxy+cy^2)^s}$		$\displaystyle\sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq 0}} \frac{\Im(\tau)^s}{\lvert m\tau+n\rvert^{2s}}$

Kronecker limits for $\mathbb{Q}(\sqrt{D})$ $(D < 0)$

$$\zeta(s, \mathscr{A}) = \frac{1}{w_D}\left(\frac{2}{\sqrt{|D|}}\right)^s E(\tau_{\mathscr{A}}, s)$$

$$\mathscr{A} \leftrightarrow \begin{matrix}\text{reduced}\\(a,b,c)\end{matrix} \leftrightarrow \tau_{\mathscr{A}} = \frac{-b+\sqrt{D}}{2a}$$

- For fixed $\tau \in \mathfrak{h}$, the Laurent expansion of E at $s = 1$ is:

$$\boxed{E(\tau, s) = \frac{\pi}{s-1} + \pi\left(\frac{\pi}{3}\Im(\tau) - \log \Im(\tau) + \mathcal{U}(\tau)\right) + 2\pi\big(\gamma - \log 2\big) + O(s-1)}$$

where:

$$\mathcal{U}(\tau) := 4\sum_{n \geq 1}\left(\sum_{d \mid n}\frac{1}{d}\right)\frac{\cos(2\pi n\, \Re(\tau))}{e^{2\pi n\, \Im(\tau)}} \qquad \left(= -2\log(|\eta(\tau)|^2) - \frac{\pi}{3}\Im(\tau)\right)$$

Kronecker's (first) limit formula

$$\mathfrak{K}(\mathscr{A}) = \underbrace{\frac{\pi}{3}\Im(\tau_{\mathscr{A}}) - \log \Im(\tau_{\mathscr{A}}) + \mathcal{U}(\tau_{\mathscr{A}})}_{\mathscr{A}\text{-dependent term}} \underbrace{-\frac{1}{2}\log|D|}_{D\text{-dependent}} \underbrace{+2\gamma - \log 2}_{\text{constant term}}$$

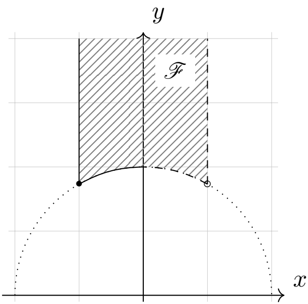
Duke's equidistribution theorem

Theorem (Duke, 1988)

$\Lambda_D = \{\tau_{\mathscr{A}} \mid \mathscr{A} \in \mathrm{C}\ell(D)\}$ *is equidistributed in* $\mathscr{F}$.

If $f : \mathscr{F} \to \mathbb{C}$ is Riemann-integrable, then:

$$\lim_{D\to-\infty} \frac{1}{h(D)} \sum_{\mathscr{A}\in\mathrm{C}\ell(D)} f(\tau_{\mathscr{A}}) = \int_{\mathscr{F}} f(z)\,\mathrm{d}\mu$$



$$z = x + iy$$

$$\mathrm{d}\mu = \frac{3}{\pi}\frac{\mathrm{d}x\mathrm{d}y}{y^2}$$

(Normalized hyperbolic area element)

W. Duke

Proof of Theorem 1

KLF: $\mathfrak{K}(\mathscr{A}) = \frac{\pi}{3}\Im(\tau_{\mathscr{A}}) - \log \Im(\tau_{\mathscr{A}}) + \mathcal{U}(\tau_{\mathscr{A}}) - \frac{1}{2}\log|D| + (2\gamma - \log 2)$

- $\int_{\mathscr{F}} \mathcal{U}(z)\,\mathrm{d}\mu = 0.000151\ldots$
- $\int_{\mathscr{F}} \log(y)\,\mathrm{d}\mu = 0.952984\ldots$
- $\int_{\mathscr{F}} \left(\log^{+}|j(z)| - 2\pi y\right)\mathrm{d}\mu = -0.068692\ldots$

$$\frac{1}{h(D)} \sum_{\mathscr{A}\in \mathrm{C}\ell(D)} \log \Im(\tau_{\mathscr{A}})$$

is **hard** without Duke's theorem!

$\Downarrow$ (by Duke's theorem)

$$\gamma_{\mathbb{Q}(\sqrt{D})} = \frac{1}{6}\left(\frac{1}{h(D)} \sum_{\mathscr{A}\in \mathrm{C}\ell(D)} \log^{+}|j(\tau_{\mathscr{A}})|\right) - \frac{1}{2}\log|D| + C' + o(1)$$

$\Downarrow$

$$\frac{L'}{L}(1,\chi_D) = \frac{1}{6}\,\mathrm{ht}(j(\tau_D)) - \frac{1}{2}\log|D| + C + o(1)$$

□

$C = -1.057770\ldots$

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Main theorem

Theorem 2 (T., 2020+)[†]

$$\text{(weak) } \textit{uniform abc} \implies \limsup_{D\to-\infty} \frac{\frac{L'}{L}(1,\chi_D)}{\log|D|} = 0$$

- **Algebraic:** $\limsup_{D\to-\infty} \frac{\mathrm{ht}(j(\tau_D))}{3\log|D|} \overset{\text{U-}abc}{\leq} 1$ [†](Granville–Stark, 2000)
- **Analytic:** $\limsup_{D\to-\infty} \frac{\mathrm{ht}(j(\tau_D))}{3\log|D|} \geq 1$ (T., 2020+) [unconditional!]

MAIN COROLLARY:

weak U-abc + Prop 1 $\Rightarrow$

$$\max\{\beta\in\mathbb{R} \mid L(\beta,\chi_D)=0\} < 1 - \frac{\sqrt{5}\varphi + o(1)}{\log|D|}$$

The summation $\sum_{\varrho(\chi)} \frac{1}{\varrho}$

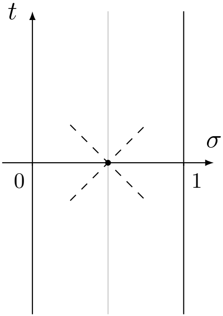
Let $D < 0$ be a fundamental discriminant.

- Classical formula (Functional Eq. + Hadamard product)

$$\frac{L'}{L}(s,\chi_D) = \left(\sum_{\varrho(\chi_D)} \frac{1}{s-\varrho} \right) - \frac{1}{2}\log\left(\frac{|D|}{\pi}\right) - \frac{\Gamma'}{\Gamma}\left(\frac{s+1}{2}\right)$$

- By the reflection formula:

$$L(\varrho,\chi) = 0 \implies \begin{cases} \varrho,\ 1-\overline{\varrho} \ \text{ zeros of } \ L(s,\chi) \\ \overline{\varrho},\ 1-\varrho \ \text{ zeros of } \ L(s,\overline{\chi}) \end{cases}$$



- Hence:

$$\boxed{\sum_{\varrho(\chi_D)} \frac{1}{\varrho} = \frac{1}{2}\log|D| + \frac{L'}{L}(1,\chi_D) - \frac{1}{2}\left(\gamma + \log\pi\right)}$$

Pairing-up zeros (1/2)

In general, writing ($\varrho \in$ critical strip, $\varepsilon > 0$):

$$\Pi_\varepsilon(\varrho) := \frac{1}{\varrho+\varepsilon} + \frac{1}{\overline{\varrho}+\varepsilon} + \frac{1}{1-\varrho+\varepsilon} + \frac{1}{1-\overline{\varrho}+\varepsilon} \quad \underline{\text{(pairing-up)}}$$

we get:

$$\boxed{\sum_{\varrho(\chi_D)} \frac{\Pi_{\sigma-1}(\varrho)}{4} = \frac{1}{2}\log|D| + \frac{L'}{L}(\sigma,\chi_D) - \frac{1}{2}\left(-\frac{\Gamma'}{\Gamma}\left(\frac{\sigma+1}{2}\right) + \log\pi\right)}$$

- Goal: Estimate Π_0 in the critical strip
- Idea: Perturb ε in Π_ε

Lemma 1 (The pairing-up lemma)

i *For every $s \in \mathfrak{S}$, we have:*

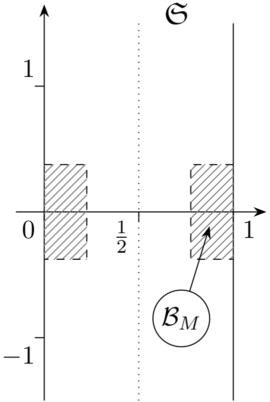
$$\Pi_0(s) > \frac{\Pi_{\varphi-1}(s)}{2\varphi - 1} \quad \left(w/\ \varphi = \frac{1+\sqrt{5}}{2} \right)$$

ii *Take $0 < \varepsilon < 1$, $M \geq 2$, and consider*

$$\mathcal{B}_M := \left\{ s \in \mathfrak{S} \;\middle|\; \sigma > 1 - \frac{1}{M},\ |t| < \frac{1}{\sqrt{M}} \right\}$$

Then, in $\mathfrak{S} \setminus \big(\mathcal{B}_M \cup (1 - \mathcal{B}_M)\big)$, we have:

$$|\Pi_0(s) - \Pi_\varepsilon(s)| < 5M\varepsilon\, \Pi_\varepsilon(s)$$



Proof of Main Corollary

Proposition 1

For any non-trivial zero ϱ of $L(s,\chi_D)$, we have:

$$\Re\left(\frac{1}{1-\varrho}\right) < \left(1-\frac{1}{\sqrt{5}}\right)\frac{1}{2}\log|D| + \frac{L'}{L}(1,\chi_D) + (\sqrt{5}\varphi - 1)$$

- Using that **<u>weak U-abc</u>** + Thm 1 $\implies \limsup_{D\to-\infty} \frac{\frac{L'}{L}(1,\chi_D)}{\log|D|} \le 0$, if $\beta \in [0,1[$ is a zero of $L(s,\chi_D)$, then

$$\frac{1}{1-\beta} < \left(\left(1-\frac{1}{\sqrt{5}}\right)\frac{1}{2} + o(1)\right)\log|D|.$$

- Since $((1-\frac{1}{\sqrt{5}})\frac{1}{2})^{-1} = \sqrt{5}\varphi$, rearranging yields $\boxed{\beta < 1 - \frac{\sqrt{5}\varphi + o(1)}{\log|D|}}$. □

Isolating the Siegel zero (1/2)

Using the *second* pairing-up lemma:

Proposition 2 (T., 2020+)

Suppose that for $\chi \pmod q$ primitive, $L(s,\chi)$ has no zeros (apart from possible Siegel zeros $\beta = \beta_\chi$) in the region

$$\left\{ s \in \mathbb{C} \;\middle|\; \sigma \geq 1 - \frac{1}{f(q)},\ |t| \leq \frac{1}{\sqrt{f(q)}} \right\},$$

for some $f : \mathbb{Z}_{\geq 2} \to \mathbb{R}$ with $2 \leq f(q) \ll \log q$. Then:

$$\Re\left(\frac{L'}{L}(1,\chi)\right) = \frac{1}{1-\beta} + O\left(\sqrt{f(q)\log q}\right)$$

- Classical ZFR: $f(q) = O(\log q)$.

$$\text{“no Siegel zeros”} \iff \frac{L'}{L}(1,\chi_D) \ll \log|D|$$

Isolating the Siegel zero (2/2)

An integer q is *k-smooth* if all its prime factors are $\leq k$.

Chang's zero-free regions (2014)

For $\chi \pmod{q}$ primitive, $L(s,\chi)$ has no zeros (apart from possible Siegel zeros) in the region

$$\left\{ s \in \mathbb{C} \;\middle|\; \sigma \geq 1 - \frac{1}{f(q)},\ |t| \leq 1 \right\},$$

where $f : \mathbb{Z}_{\geq 2} \to \mathbb{R}$ satisfies:

$$f(q) = o(\log q) \text{ for } q^{o(1)}\text{-smooth moduli}$$

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- Chang's ZFR + Proposition 2:

$$\boxed{\frac{L'}{L}(1,\chi_D) = \frac{1}{1-\beta_D} + O\Big(\sqrt{f(|D|)\,\log|D|}\Big),}$$

where $\beta_D := \max\{\beta \in \mathbb{R} \mid L(\beta,\chi_D) = 0\}$.

Proof of Main Theorem

$$\boxed{\frac{L'}{L}(1,\chi_D) = \frac{1}{1-\beta_D} + O\Big(\sqrt{f(|D|)\,\log|D|}\Big)}$$

- Since $\dfrac{1}{1-\beta_D} > 0$, and $\sqrt{f(|D|)\,\log|D|} = o(\log|D|)$ for $|D|^{o(1)}$-smooth fundamental discriminants, it follows that

$$\limsup_{\substack{D\to-\infty \\ |D|^{o(1)}\text{-smooth}}} \frac{\frac{L'}{L}(1,\chi_D)}{\log|D|} \geq 0. \qquad \square$$

Corollary (strong "no Siegel zeros")

Assume weak uniform *abc*. For any $A>0$, as $D\to-\infty$ through $|D|^{o(1)}$-smooth discriminants, all but finitely many D satisfy

$$\max\{\beta\in\mathbb{R} \mid L(\beta,\chi_D)\} < 1 - \frac{A}{\log|D|}.$$

Upper bounds for $h(D)$

For fundamental discriminants $D < 0$ we have:

$$\frac{3}{\sqrt{5}} \log|D| + O(1) \le \mathrm{ht}(j(\tau_D)) \overset{*}{\le} (1 + o(1))\, 3\log|D|$$

$$\boxed{(1 + o(1))\, \frac{\pi}{3} \frac{\sqrt{|D|}}{\log|D|} \sum_{(a,b,c)} \frac{1}{a} \overset{*}{\le} h(D) \le \left(\sqrt{5} + O\left(\frac{1}{\log|D|}\right)\right) \frac{\pi}{3} \frac{\sqrt{|D|}}{\log|D|} \sum_{(a,b,c)} \frac{1}{a}}$$

where:

- starred ineq.s "$\overset{*}{\le}$" are conditional on the weak uniform ABC.
- if D is $|D|^{o(1)}$-smooth, then starred ineq.s become (asymptotic) *equalities*.
- GRH $\implies$ asymptotic starred *equalities* for all D, with error term a factor of $O(\frac{\log\log|D|}{\log|D|})$. (E.g., $\mathrm{ht}(j(\tau_D)) = 3\log|D| + O(\log\log|D|)$ under GRH).

ありがとう
ございました

References

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