Morse theory, Graphs, and Loop spaces

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In these lectures I will describe algebraic topological aspects of Morse theory, and applications both to the topology of finite dimensional manifolds, and to certain infinite dimensional manifolds such as loop spaces. A particular goal will be the description of the "string topology" operations of Chas and Sullivan in Morse theoretic terms, putting this theory in the context of topological and conformal field theory. I will also discuss relations with Gromov-Witten theory of the cotangent bundle. Topics to be covered include the following.

1. *The "flow category" of a Morse function.*

   The objects of this category are the critical points of a Morse function, and the morphisms are the compactified moduli spaces of gradient flow lines. We will discuss work of Cohen, Jones, and Segal that describes the topology of a compact manifold in terms of this category, and applications to infinite dimensions.

2. *The moduli space of graph flows and cohomology operations.*

   We will describe work of Betz and Cohen that uses "gradient graphs in a manifold" to describe classical cohomology operations such as Poincare duality, Steenrod operations, and characteristic classes. We will also discuss more recent work of Cohen and Norbury drawing analogies with constructions in Gromov-Witten theory. In particular we study the moduli space of metrics in the graphs, and produce a virtual fundamental class, using algebraic topological, rather than algebraic geometric methods. We will discuss the field theoretic properties of these constructions. We discuss and relate this point of view to that of Fukaya.

3. *Spaces of graphs* We recall work of Penner, Strebel, and Kontsevich on how spaces of "fat graphs" are used as simplicial models for moduli spaces of Riemann surfaces. We use them to study spaces of holomorphic maps and smooth maps of Riemann surfaces in a manifold.

4. *String topology - a Morse theoretic viewpoint.*

   We review the basics of the string topology operations recently defined by Chas and Sullivan. We use moduli spaces of fat graphs to give a description of the these operations in terms of graph flows of the Dirichlet energy functional on the loop space of a manifold. We will discuss field theoretic properties of these operations. We relate these moduli spaces of gradient graph flows in M to moduli spaces of holomorphic curves in the cotangent bundle of M. This will allow us to discuss the relation between the string topology of M and the Gromov-Witten theory of T^*M.

References

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