The Morse complex for infinite dimensional manifolds

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Abstract. Let (M, g) be a Riemannian manifold. The Morse complex approach to Morse theory consists in associating a chain complex to a Morse function $f : M \to \mathbb{R}$, by looking at the intersections of the unstable and stable manifolds of critical points of f, with respect to the negative gradient flow of f. This approach, originally developed for compact manifolds, easily extends to functions defined on an infinite dimensional Hilbert manifold, provided that the Palais-Smale condition holds and that all critical points have finite Morse index. In this situation one finds an alternative picture of infinite dimensional Morse theory, as developed by Palais and by Smale in the sixties.

What is more relevant, this approach has been successfully used also in some cases involving critical points of infinite Morse index and coindex. The standard approach fails here, because such critical points are homotopically invisible, the reason being that the infinite dimensional ball is retractable onto its boundary. The unstable and the stable manifolds of critical points are infinite dimensional, but they may still have finite dimensional intersection, so the Morse complex approach may work in this case. Indeed, this is the idea behind Floer homology, in which actually the gradient equation (a well-posed ODE on an infinite dimensional manifold) is replaced by the gradient equation with respect to a metric which is not complete on the function space where f is differentiable, which has however the nice property of being an elliptic PDE.

In these lectures we will show how to build the Morse complex for smooth functions on complete Hilbert manifolds, possibly with critical points with infinite Morse indices. More precisely, we will give an answer to the following questions: (i) when are the intersections between the unstable and stable manifolds finite dimensional? (ii) when are they orientable? (iii) when are they pre-compact? (iv) how does one prove the boundary property? (v) how can the resulting homology be computed? The main extra structure needed in this theory is the presence of a subbundle of the tangent bundle

of M, somehow compatible with the gradient flow of f. More generally, one can deal with an essential subbundle, i.e. a subbundle defined only up to compact perturbations. Relevant tools are the study of the Fredholm property for ordinary differential operators on Hilbert spaces, and the study of some infinite dimensional Grassmannians.

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