Divergent series: past, present, future …[*]

Christiane ROUSSEAU

Département de mathématiques et de statistique and CRM
Université de Montréal
C.P. 6128, Succursale Centre-ville, Montréal (Qué.), H3C 3J7, Canada

December 17, 2013

Abstract

The present paper presents some reflections of the author on divergent series and their role and place in mathematics over the centuries. The point of view presented here is limited to differential equations and dynamical systems.

1 Introduction

In the past, divergent series have played a central role in mathematics. Mathematicians like for instance Lacroix, Fourier, Euler, Laplace, etc. have used them extensively. Nowadays they play a marginal role in mathematics and are often not mentioned in the standard curriculum. A number of mathematicians and most students are not aware that they can be of any use.

<p align="center">Where was the turn?</p>

According to [2], the turn occured in the time of Cauchy and Abel, when the need was felt to construct analysis on absolute rigor.

So let us go back a little and see what Cauchy and Abel have said of divergent series.

Cauchy (Preface of "Analyse mathématique", 1821): *"J'ai été forcé d'admettre diverses propositions qui paraîtront peut-être un peu dures. Par exemple qu'une série divergente n'a pas de somme…"* ("I have been forced to admit some propositions which will seem, perhaps, hard to accept. For instance that a divergent series has no sum …")

Cauchy made one exception: he justified rigorously the use of the divergent Stirling series to calculate the Γ-function. We will explain below the kind of argument he made when looking at the example of the Euler differential equation.

Abel (Letter to Holmboe, January 16 1826): *"Les séries divergentes sont en général quelque chose de bien fatal et c'est une honte qu'on ose y fonder aucune démonstration. On peut démontrer tout ce qu'on veut en les employant, et ce sont elles qui ont fait tant de malheurs et qui ont enfanté tant de paradoxes. ...Enfin mes yeux se sont dessillés d'une manière frappante, car à l'exception des cas les plus simples, par exemple les séries géométriques, il ne se trouve dans les mathématiques presque aucune série infinie dont la somme soit déterminée d'une manière rigoureuse, c'est-à-dire que la partie la plus essentielle des mathématiques est sans fondement. Pour la plus grande partie les résultats sont justes il est vrai, mais c'est là une chose bien étrange. Je m'occupe à en chercher la raison, problème très intéressant."* ("Divergent series are, in general, something terrible and it is a shame to base any proof on them. We can prove anything by using them and they have caused so much misery and created so many paradoxes. .... Finally my eyes were suddenly opened since, with the exception of the simplest cases, for instance the geometric series, we hardly find, in mathematics, any infinite series whose sum may be determined in a rigorous fashion, which means the most essential part of mathematics has no foundation. For the most part, it is true that the results are correct, which is very strange. I am working to find out why, a very interesting problem.")

The author was struck by this citation when she first read it 17 years ago and this citation has followed her for her whole career. In her point of view, this citation contains the past, present and future of divergent series in mathematics.

The past: As remarked by Abel, divergent series occur very often in many natural problems of mathematics and physics. Their use has permitted to do successfully a lot of numerical calculations. One example of this is the computation by Laplace of the secular perturbation of the orbit of the Earth around the Sun due to the attraction of Jupiter. The calculations of Laplace are verified experimentally, although the series he used were divergent.

The present: In the 20-th century divergent series have occupied a marginal place. However it is during the same period that we have learnt to justify rigorously their use, answering at least partially Abel's question. In the context of differential equations the Borel summation, generalized by Écalle and others to multi-summability, give good results (see for instance [1], [2], [3] [8] and [12]).

The future: Why do so many important problems of mathematics lead to divergent series (see for instance [5])? What is the meaning of a series being divergent?

We will illustrate all this on the example of the Euler differential equation:

$$x^2 y' + y = x. \tag{1.1}$$

As this is a short paper the list of references is by no means exhaustive.

2 The past

The Euler differential equation (1.1) has the formal solution

$$\hat{f}(x) = \sum_{n \geq 0} (-1)^n n! x^{n+1}, \tag{2.1}$$

which is divergent for all nonzero values of x.

On the other hand it is a linear differential equation whose solution can be found by quadrature:

$$f(x) = e^{\frac{1}{x}} \int_0^x \frac{e^{-\frac{1}{z}}}{z} dz. \tag{2.2}$$

The integral is convergent for $x > 0$ and hence yields a solution of (1.1). We can rewrite this solution as

$$f(x) = \int_0^x \frac{e^{\frac{1}{x}-\frac{1}{z}}}{z} dz, \tag{2.3}$$

in which we make the change of coordinate $\frac{\zeta}{x} = \frac{1}{z} - \frac{1}{x}$. This yields

$$f(x) = \int_0^{+\infty} \frac{e^{-\frac{\zeta}{x}}}{1+\zeta} d\zeta. \tag{2.4}$$

The integral is convergent for $x \geq 0$, so the solution $f(x)$ is well defined for $x \geq 0$ and moreover $f(0) = 0$.

What is now the link between the divergent power series $\hat{f}(x)$ and the function $f(x)$? We will show that the difference between $f(x)$ and a partial sum

$$f_k(x) = \sum_{n=0}^{k-1} (-1)^n n! x^{n+1} \tag{2.5}$$

is smaller than the first neglected term (this part has been inspired by [12]). This is exactly as in the Leibniz criterion for alternating series.

Proposition 2.1 *For any $x \geq 0$*

$$|f(x) - f_k(x)| \leq k! x^{k+1}. \tag{2.6}$$

PROOF. We use the following formula which is easily checked

$$\frac{1}{1+\zeta} = \sum_{n=0}^{k-1} (-1)^n \zeta^n + (-1)^k \frac{\zeta^k}{1+\zeta}. \tag{2.7}$$

Then

$$\begin{aligned}
f(x) &= \int_0^{+\infty} e^{-\frac{\zeta}{x}} \left(\sum_{n=0}^{k-1} (-1)^n \zeta^n + (-1)^k \frac{\zeta^k}{1+\zeta} \right) d\zeta \\
&= \sum_{n=0}^{k-1} \int_0^{+\infty} (-1)^n \zeta^n e^{-\frac{\zeta}{x}} d\zeta + \int_0^{+\infty} (-1)^k \frac{\zeta^k e^{-\frac{\zeta}{x}}}{1+\zeta} d\zeta.
\end{aligned} \tag{2.8}$$

Using the following formula

$$\Gamma(n+1) = n! = \int_0^\infty z^n e^{-z} dz, \tag{2.9}$$

which implies

$$\int_0^\infty \zeta^n e^{-\frac{\zeta}{x}} d\zeta = n! x^{n+1}, \tag{2.10}$$

this yields

$$\begin{aligned} f(x) &= \sum_{n=0}^{k-1} (-1)^n n! x^{n+1} + \int_0^{+\infty} (-1)^k \frac{\zeta^k e^{-\frac{\zeta}{x}}}{1+\zeta} d\zeta \\ &= f_k(x) + R_k(x), \end{aligned} \tag{2.11}$$

where $R_k(x)$, the remainder, is the difference between $f(x)$ and the partial sum $f_k(x)$ of the power series. The result follows as

$$\begin{aligned} |R_k(x)| &= \int_0^\infty \frac{\zeta^k e^{-\frac{\zeta}{x}}}{1+\zeta} d\zeta \\ &\leq \int_0^\infty \zeta^k e^{-\frac{\zeta}{x}} d\zeta \\ &= k! x^{k+1}. \end{aligned} \tag{2.12}$$

$$\square$$

The argument given here, which justifies rigorously the use of the divergent series in numerical calculations, is very similar to the argument made by Cauchy for the use of the Stirling series. In particular Cauchy used the formula (2.7).

If we use the power series to approximate the function $f(x)$, how good is the approximation?

We encounter here the main difference between convergent and divergent series. With convergent series, the more terms we take in the partial sum, the better the approximation. This is not the case with divergent series. Indeed, if we take x fixed in (2.1) the general term tends to ∞. So we are better to take the partial sum for which the first neglected term is minimum. $k! x^{k+1}$ is minimum when $x \sim \frac{1}{k}$. In that case $\frac{1}{x} \sim k$. We use Stirling formula to approximate $k!$ for k large:

$$k! = k(k-1)! = k\Gamma(k) \sim \sqrt{2\pi} k^{k+\frac{1}{2}} e^{-k}. \tag{2.13}$$

This gives us

$$|R_k(x)| \leq \sqrt{2\pi} \frac{e^{-k}}{\sqrt{k}} \sim \sqrt{2\pi x} e^{-\frac{1}{x}}, \tag{2.14}$$

which is exponentially small for x small. Not only have we bounded the error made when approximating the function by the partial sum of the power series, but this error is exponentially small for small x, i.e. very satisfactory on the numerical point of view.

The phenomenon we have described here is not isolated and explains the successes encountered when using divergent series in numerical approximations.

3 The present

Looking at what we have done with the Euler equation, someone can have the impression we have cheated. Indeed we have chosen a linear differential equation, thus allowing us to

construct by quadrature a function which is solution of the differential equation. But what about the general case?

In general, once we have a formal solution by means of a power series, we use a theory of resummation to find a function which is a solution. In this paper we will focus on the Borel method of resummation, also called 1-summability. We start with the properties that must satisfy an adequate theory of summability:

Properties of a good method of resummation (see for instance [2]): we consider a series $\sum_{n=0}^{\infty} a_n$, to which we want to associate a number S called its sum:

(1) If $\sum_{n=0}^{\infty} a_n$ is convergent, then S should be the usual sum of the series.

(2) If $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are summable with respective sums S and S' then $\sum_{n=0}^{\infty}(a_n + Cb_n)$ is summable with sum $S + CS'$.

(3) If $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are absolutely summable with respective sums S and S', then $\sum_{n=0}^{\infty} c_n$, where $c_n = \sum_{i=0}^{n} a_i b_{n-i}$, is absolutely summable with sum SS'.

(4) If $\sum_{n=0}^{\infty} a_n$ is summable with sum S, then $\sum_{n=1}^{\infty} a_n$ is summable with sum $S - a_0$.

(5) (In the context of differential equations) If $\sum_{n=0}^{\infty} a_n x^n$ is summable with sum $f(x)$ then $\sum_{n=1}^{\infty} n a_n x^{n-1}$ is summable with sum $f'(x)$.

The Borel method of resummation for a series: we present it in a way which proves at the same time the property (1): the idea is to take a convergent series $\sum_{n=0}^{\infty} a_n$ and to write its sum $S = \lim_{k \to \infty} \sum_{n=0}^{k} a_n$ in a different way. For this purpose we use (2.9) and we write

$$
\begin{aligned}
S &= \sum_{n=0}^{\infty} a_n \\
&= \sum_{n=0}^{\infty} \frac{a_n}{n!} \, n! \\
&= \sum_{n=0}^{\infty} \frac{a_n}{n!} \int_{0}^{\infty} z^n e^{-z} dz \\
&= \int_{0}^{\infty} \left(\sum_{n=0}^{\infty} \frac{a_n}{n!} z^n \right) e^{-z} dz.
\end{aligned}
\tag{3.1}
$$

Definition 3.1 A series $\sum_{n=0}^{\infty} a_n$ is *Borel-Summable* if the series $\sum_{n=0}^{\infty} \frac{a_n}{n!} z^n$ has a nonzero radius of convergence, if it can be extended along the positive real axis and if the integral

$$
\int_{0}^{\infty} \left(\sum_{n=0}^{\infty} \frac{a_n}{n!} z^n \right) e^{-z} dz
\tag{3.2}
$$

is convergent with value S. We call S the *Borel sum* of the series.

Example: In the case of the solution of the Euler differential equation we have $a_n = (-1)^n n! x^{n+1}$. Then

$$\sum_{n=0}^{\infty} \frac{a_n}{n!} z^n = x \sum_{n=0}^{\infty} (-1)^n (xz)^n = \frac{x}{1+xz}. \tag{3.3}$$

Hence

$$\int_0^{\infty} \left(\sum_{n=0}^{\infty} \frac{a_n}{n!} z^n \right) e^{-z} dz = \int_0^{\infty} \frac{xe^{-z}}{1+xz} dz = \int_0^{\infty} \frac{e^{-\frac{\zeta}{x}}}{1+\zeta} d\zeta, \tag{3.4}$$

which is (2.4).

We adapt the definition of Borel-summability to power series in a manner that will allow to use it for analytic extension.

Definition 3.2   1. A power series $\sum_{n=0}^{\infty} a_n x^{n+1}$ is 1-summable (Borel-summable) in the direction d, where d is a half-line from the origin in the complex plane if the series $\sum_{n=0}^{\infty} a_n \frac{\zeta^n}{n!}$ has a nonzero radius of convergence and can be extended along the half line d, and if the integral

$$\int_d \left(\sum_{n=0}^{\infty} \frac{a_n \zeta^n}{n!} \right) e^{-\frac{\zeta}{x}} d\zeta \tag{3.5}$$

is convergent with value $S(x)$. We call S the sum of the series.

2. A power series $\sum_{n=0}^{\infty} a_n x^{n+1}$ is 1-summable if it is 1-summable in all directions d except a finite number of exceptional directions.

Example: The solution of the Euler differential equation is 1-summable in all directions except the direction $\mathbb{R}^-$. The problem in the direction $\mathbb{R}^-$ comes from the singularity at $\zeta = -1$ in (2.4) or (3.4).

A theory of resummation is useful if there are theorems associated to it. For instance, for the Borel summation, let us cite this theorem of Borel:

Theorem 3.3 *[2] We consider an algebraic differential equation*

$$F(x, y, y', \dots y^{(m)}) = 0, \tag{3.6}$$

where F is a multivariate polynomial. If $\hat{f}(x) = \sum_{n=0}^{\infty} a_n x^n$ is a formal solution of (3.6) and $\hat{f}$ is absolutely Borel-summable with Borel sum $f(x)$, then $f(x)$ is solution of the differential equation (3.6) and has the asymptotic expansion $\hat{f}(x) = \sum_{n=0}^{\infty} a_n x^n$.

Remark 3.4   (1) The sums of a 1-summable series in the different directions d give functions which are analytic extensions one of the other as long as we move the line d continuously through directions in which the series is 1-summable. This yields a function defined on a sector with vertex at the origin. More details can be found for instance in [1], [11] and [12].

(2) The Borel sum of a divergent power series can never be uniform in a punctured neighborhood of the origin. It is necessarily ramified. This is what is known in the literature as the *Stokes phenomenon.*

(3) If a series $\sum_{n=0}^{\infty} a_n x^{n+1}$ has a radius of convergence equal to r and its sum is a function $f(x)$ for $|x| < r$, then a theorem of complex analysis states that the function $f(x)$ has a least one singularity on the circle $|x| = r$. The idea of Borel is that a divergent series is a series with radius of convergence $r = 0$. Hence we have at least one singularity hidden in one direction: for the Euler equation this is the direction $\mathbb{R}^-$. The operation

$$\hat{f}(x) = \sum_{n=0}^{\infty} a_n x^{n+1} \qquad \mapsto \qquad \mathcal{B}(\hat{f}) = \sum_{n=0}^{\infty} \frac{a_n \zeta^n}{n!}, \tag{3.7}$$

sends the singularity at a finite distance. $\mathcal{B}(\hat{f})$ is called the *Borel transform* of $\hat{f}$.

The 1-summability and its extensions have been extensively studied during the 20-th century. Extensions of the notion of 1-summability have been obtained by allowing ramifications $x = \xi^m$. Then 1-summability in x corresponds to m-summability in ξ. The notion of multi-summability has also been introduced: a series $\hat{f}$ is multi-summable if it is a finite sum $\hat{f} = \hat{f}_1 + \cdots + \hat{f}_n$, each $\hat{f}_i$ being m_i-summable. Explicit criteria allow to decide a priori that some systems of differential equations have multi-summable solutions in the neighborhood of certain singular points (see for instance [1], [3], [8], [11], [12]).

4   The future

Let us now look at a generalized Euler equation

$$x^2 y' + y = g(x). \tag{4.1}$$

For almost all functions $g(x)$ analytic in the neighborhood of the origin and such that $g(0) = 0$ the formal solution of (4.1) vanishing at the origin is given by a divergent series. Only in very special cases the equation (4.1) has an analytic solution at the origin.

Example: $f(x) = x$ is the analytic solution of

$$x^2 y' + y = x + x^2. \tag{4.2}$$

What is the difference between the equations (1.1) and (4.2)? To understand we will apply successively the two steps:

- complexify: we will allow $x \in \mathbb{C}$;

- unfold: $x = 0$ is a double singular point of each equation.

Complexification: we consider a function $f(x)$ which is the Borel sum of a solution of (4.1) and we consider its analytic extension when we turn around the origin. The function $f(x)$ is (Figure 1):

Figure 1: The domain of $f(x)$

- uniform for (4.2);

- ramified for (1.1) and generically ramified for a solution of (4.1). The two branches differ by a multiple of $e^{\frac{1}{x}}$ (which is a multiple of a solution of the homogeneous equation).

How to understand that generically we should have ramification? To answer we unfold and embed (4.1) into

$$(x^2 - \epsilon)y' + y = g_\epsilon(x), \tag{4.3}$$

with $\epsilon \in \mathbb{R}^+$. We will limit here our discussion to $\epsilon > 0$, although all complex values of ϵ are of interest. As solutions of the homogeneous equation associated to (4.3) are given by $C\left(\frac{x+\sqrt{\epsilon}}{x-\sqrt{\epsilon}}\right)^{\frac{1}{2\sqrt{\epsilon}}}$, the local model of solutions at $x_1 = \sqrt{\epsilon}$ is given by

$$y_1(x) = h_1(x) + C_1\left(\frac{x+\sqrt{\epsilon}}{x-\sqrt{\epsilon}}\right)^{\frac{1}{2\sqrt{\epsilon}}}, \tag{4.4}$$

with $h_1(x)$ analytic. Hence the solution $h_1(x)$ (corresponding to $C_1 = 0$) is the unique solution which is analytic and bounded at $x = \sqrt{\epsilon}$. Similarly the local model for solutions at $x_2 = -\sqrt{\epsilon}$ is given by

$$y_2(x) = h_2(x) + C_2\left(\frac{x+\sqrt{\epsilon}}{x-\sqrt{\epsilon}}\right)^{\frac{1}{2\sqrt{\epsilon}}}. \tag{4.5}$$

We now have two cases:

(1) If $\frac{1}{2\sqrt{\epsilon}} \notin \mathbb{N}$, then $h_2(x)$ is analytic. As $C_2\left(\frac{x+\sqrt{\epsilon}}{x-\sqrt{\epsilon}}\right)^{\frac{1}{2\sqrt{\epsilon}}}$ is ramified at $x = -\sqrt{\epsilon}$ for $C_2 \neq 0$, all solutions but $h_2(x)$ are ramified. It is of course exceptional that the extension of $h_1(x)$ at $-\sqrt{\epsilon}$ be exactly the solution $h_2(x)$ and, generically, we should expect that the analytic extension of h_1 is $h_2(x) + C_2\left(\frac{x+\sqrt{\epsilon}}{x-\sqrt{\epsilon}}\right)^{\frac{1}{2\sqrt{\epsilon}}}$ with $C_2 \neq 0$. Hence the extension of $h_1(x)$ should be ramified at $x = -\sqrt{\epsilon}$. If this ramification holds till the limit $\epsilon = 0$ we get Figure 1(a).

(2) If $\frac{1}{2\sqrt{\epsilon}} \in \mathbb{N}$ we must again divide the discussion in two cases:

  (i) In the generic case $h_2(x)$ is ramified: it contains one term of the form $(x+\sqrt{\epsilon})^n \ln(x+\sqrt{\epsilon})$.

(ii) In the exceptional case $h_2(x)$ is analytic, and so are all solutions through $x = -\sqrt{\epsilon}$. Hence it is impossible for $h_1(x)$ to be ramified at $x = -\sqrt{\epsilon}$. This case is excluded in the unfolding of an equation (4.1) whose solution is ramified.

Let us now summarize the very interesting phenomenon we have discovered: if the formal solution of (4.1) is divergent, then in the unfolding:

- Necessarily $h_1(x)$ is ramified at $-\sqrt{\epsilon}$: the divergence means a form of incompatibility between the local solutions at two singular points $\pm\sqrt{\epsilon}$, which remains until the limit at $\epsilon = 0$. A start in this direction was done by Martinet [9] and continued recently for instance in [4], [7], [10].

- **Parametric resurgence phenomenon:** for sequences of values of the parameter ϵ converging to $\epsilon = 0$ (here $\frac{1}{2\sqrt{\epsilon}} \in \mathbb{N}$) the pathology of the system is located exactly at one of the singular points. Indeed, here, the only way for the system to have an incompatibility is that one of the singular points be pathologic (see for instance [10] and [13]).

We have understood why divergence is the rule and convergence the exception. These phenomena are much more general than the context of (4.1) described here. There are a few works in this direction when the divergent series is Borel-summable ([9], [7], [4]). In particular the parametric resurgence phenomenon is described in [10] and [13].

Divergent series also occur in the phenomena involving "small denominators". A source of divergence in this case is what the Russian school calls "materialization of resonances" (e.g. [6]). For instance, in the case of a fixed point of a germ of analytic diffeomorphism $(\mathbb{C}, 0) \to (\mathbb{C}, 0)$ with multiplier $e^{2\pi i \alpha}$ with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, the divergence of the linearizing series could come from the accumulation of periodic points ([14]).

5 Conclusion

Divergent series occur generically in many situations with differential equations and dynamical systems. Their divergence carries a lot of geometric information on the solutions of their equations. For instance, if the formal power series solution of the Euler equation (1.1) had been convergent, its sum could not have been ramified. The space of sums of convergent power series is not sufficiently rich to encode the rich dynamics of the solutions of differential equations, hence the divergence.

The future:

- A large program of research will consist in learning to "read" all the rich behaviour of functions defined by divergent series.

- Should divergent series occupy a more important place in mathematics?

References

[1] W. Balser, *From divergent series to analytic differential equations*, Springer Lecture Notes in Mathematics, 1582, Springer-Verlag, Berlin, (1994).

[2] E. Borel, *Leçons sur les séries divergentes*, Gauthier-Villars, 1901.

[3] J. Écalle, *Les fonctions résurgentes*, Publications mathématiques d'Orsay, 1985.

[4] A. A. Glutsyuk, Confluence of singular points and nonlinear Stokes phenomenon, *Trans. Moscow Math. Soc.*, **62** (2001), 49–95.

[5] Y. Ilyashenko, In the theory of normal forms of analytic differential equations, divergence is the rule and convergence the exception when the Bryuno conditions are violated, *Moscow University Mathematics Bulletin*, **36** (1981) 11–18.

[6] Y.S. Ilyashenko, and A.S. Pyartli, Materialization of Poincaré resonances and divergence of normalizing series, *J. Sov. Math.* **31** (1985), 3053–3092.

[7] P. Lavaurs, Systèmes dynamiques holomorphes: explosion de points périodiques paraboliques, Ph.D. Thesis, Université de Paris-Sud, 1989.

[8] M. Loday-Richaud, *Introduction à la multi-sommabilité*, Gaz. Math. No. 44 (1990), 41–63.

[9] J. Martinet, Remarques sur la bifurcation nœud-col dans le domaine complexe, *Astérisque* **150-151** (1987), 131–149.

[10] P. Mardešić, R. Roussarie and C. Rousseau, *Modulus of analytic classification for unfoldings of generic parabolic diffeomorphisms*, to appear in Moscow Mathematical Journal.

[11] J.-P. Ramis, *Les séries k-sommables et leurs applications*, Springer Lecture Notes in Physics, 126 (1980), 178–199.

[12] J.-P. Ramis, *Séries divergentes et théories asymptotiques*, Bull. Soc. Math. France 121 (1993), Panoramas et Synthèses, suppl., 74 pp.

[13] C. Rousseau, *Modulus of orbital analytic classification for a family unfolding a saddle-node*, to appear in Moscow Mathematical Journal.

[14] J.-C. Yoccoz, *Théorème de Siegel, nombres de Bruno et polynômes quadratiques*, Astérisque **231** (1995),1–88.