Divergent series past, present, future. . .

Christiane Rousseau

How many students and mathematicians have never heard of divergent series?

Preamble: My presentation of the subject and of some of its history is not exhaustive. Moreover, it is biased by my interest in dynamical systems.

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Euler:

"If we get the sum

$$1-1+1-1+1-1+\cdots$$

its only reasonable value is $\frac{1}{2}$*."*

A hundred years before Riemann, Euler had found the functional equation of the function ζ in the form

$$\frac{1^{s-1}-2^{s-1}+3^{s-1}-\cdots}{\frac{1}{1^s}-\frac{1}{2^s}+\frac{1}{3^s}-\cdots}=-\frac{(s-1)!(2^s-1)}{(2^{s-1}-1)\pi^s}\cos\frac{1}{2}s\pi$$

by "calculating", for $s\in\mathbb{N}\cup\{1/2,3/2\}$, the sums

$$1^s-2^s+3^s-4^s+5^s-6^s+\cdots$$

and

$$\frac{1}{1^s}-\frac{1}{2^s}+\frac{1}{3^s}-\frac{1}{4^s}+\frac{1}{5^s}-\frac{1}{6^s}+\cdots$$

Where is the turn?

Cauchy, Abel

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Cauchy made one exception: he justified *rigorously* the use of Stirling's divergent series in numerical computations.

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The two rules are legitimate: the first one in the theoretical researches; the second one in the numerical applications …"

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For instance if a power series is a solution of a differential equation, then its sum should be a function which is a solution of this differential equation.

Euler's differential equation

$$x^2y' + y - x = 0$$

This equation has the formal solution

$$\hat{f}(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$$

where

$$\begin{cases} a_0 = 1 \\ a_n = -na_{n-1} \end{cases}$$

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Hence,

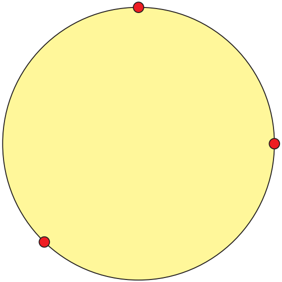
$$\hat{f}(x) = \sum_{n=0}^{\infty} (-1)^n n!\, x^{n+1}$$

Borel's vision

A convergent series

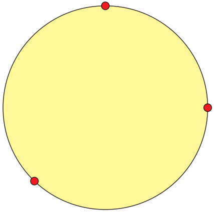
$$\sum_{n=0}^{\infty} a_n x^{n+1}$$

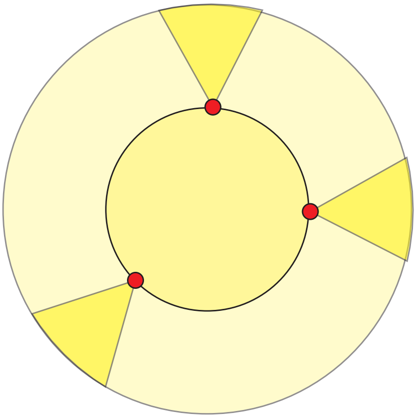
has a disk of convergence $B(0, r)$. There exists at least one singularity on the boundary of the disk of convergence



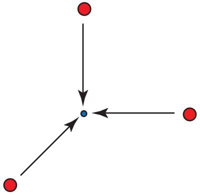
We can extend the function by turning around the singularities.
Generically this extension is ramified.

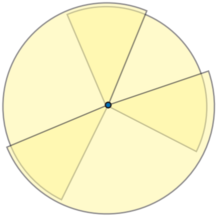
$$\log(1-x)+\log(i-x)+\log\left(e^{\frac{5\pi i}{4}}-x\right)$$





In Borel's mind, a divergent series is a convergent series on a disk of radius $r = 0$. If there are only a finite number of singularities on the boundary of the "disk", i.e. in a finite number of directions, then we can extend the function around these, and we get a function defined on sectors.





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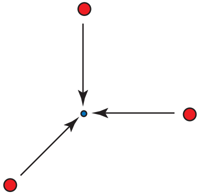

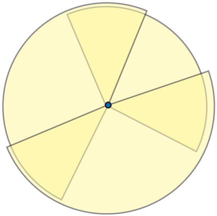

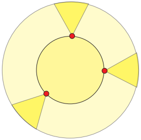

Where are hidden these singularities?

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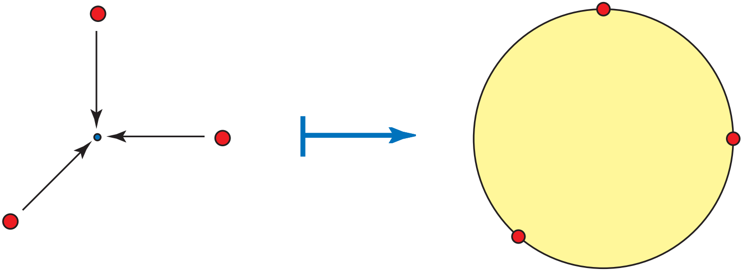
To find them we "blow-up"



Basic rules for a summation method

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- If $S = \sum_{n=0}^{\infty} a_n$ and $S' = \sum_{n=0}^{\infty} b_n$, then

$$S + cS' = \sum_{n=0}^{\infty} (a_n + cb_n)$$

- If $S = \sum_{n=0}^{\infty} a_n$ and $S' = \sum_{n=0}^{\infty} b_n$ are absolutely summable, then the product series is absolutely summable

$$S \cdot S' = \sum_{n=0}^{\infty} \left(\sum_{j+k=n} a_j b_k \right)$$

Borel summation method of divergent power series

Let us take a convergent series $\sum_{n=0}^{\infty} a_n x^{n+1}$

$$\sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=0}^{\infty} n! \, \frac{a_n x^{n+1}}{n!}$$

$$= \sum_{n=0}^{\infty} \underbrace{\left(\int_0^{\infty} y^n e^{-y} dy \right)}_{n!} \frac{a_n x^{n+1}}{n!}$$

$$= \int_0^{\infty} e^{-y} \sum_{n=0}^{\infty} \frac{a_n (xy)^n}{n!} \underbrace{xdy}_{d\zeta} \qquad \zeta = xy$$

$$= \int_0^{\infty} e^{-\frac{\zeta}{x}} \sum_{n=0}^{\infty} \frac{a_n \zeta^n}{n!} d\zeta$$

The Borel sum of a series

This suggests defining

$$\int_0^\infty e^{-\frac{\zeta}{x}} \sum_{n=0}^{\infty} \frac{a_n \zeta^n}{n!} \, d\zeta$$

as the *sum* of the series $\sum_{n=0}^{\infty} a_n x^{n+1}$, whenever this expression makes sense.

Summability on the direction d

The power series $\sum_{n=0}^{\infty} a_n x^{n+1}$ is *Borel-summable in the direction* d if

- $$\sum_{n=0}^{\infty} \frac{a_n \zeta^n}{n!} d\zeta$$

  is convergent in a disk $B(0, r)$,
- the sum of this series can be extended along the half-line d and its growth is at most exponential at infinity.

Then the sum of the series is given by

$$\sum_{0}^{\infty} a_n x^{n+1} = \int_d e^{-\frac{\zeta}{x}} \sum_{n=0}^{\infty} \frac{a_n \zeta^n}{n!} d\zeta$$

Borel summability

The power series $\sum_{n=0}^{\infty} a_n x^{n+1}$ is *Borel-summable* if it is Borel-summable in the direction d for all directions d, but a finite number.

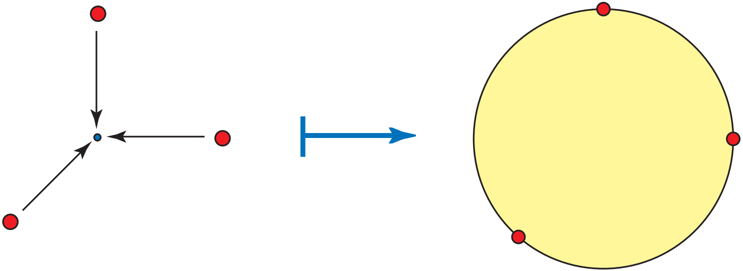
The modern presentation of this process

Three steps:

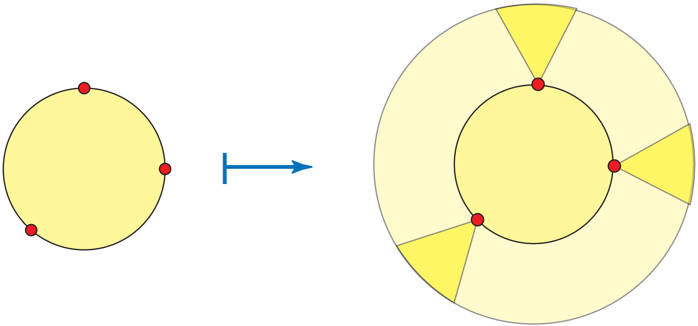
- We "blow-up"to send the hidden singularities at the singular point to a finite distance: this is done by the *Borel transform*.
- We extend along half-lines avoiding the singularities.
- We compute the sum in applying the *Laplace transform*.

Borel transform

$$\hat{f}(x) = \sum_{n=0}^{\infty} a_n x^{n+1} \qquad \mapsto \qquad \mathscr{B}(\hat{f})(\zeta) = \sum_{n=0}^{\infty} \frac{a_n}{n!} \zeta^n$$

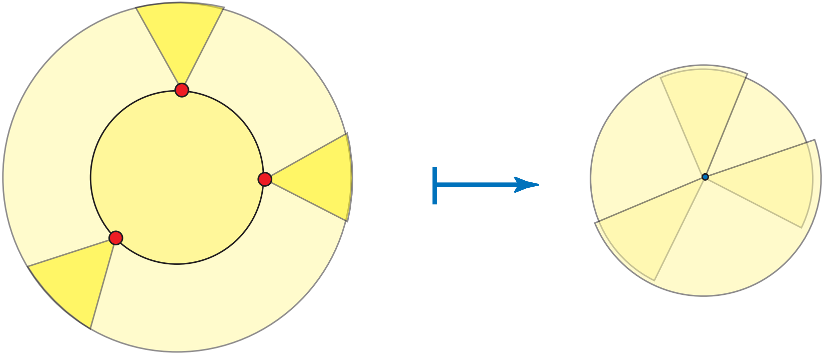


Extending along half-lines avoiding the singularities



Laplace transform

$$\mathscr{B}(\hat{f})(\zeta) \qquad \mapsto \qquad \mathscr{L}\left(\mathscr{B}(\hat{f})\right)(x) = \int_d e^{-\frac{\zeta}{x}} \mathscr{B}(\hat{f})(\zeta)\, d\zeta$$



Coming back to Euler's equation $x^2y' + y - x = 0$

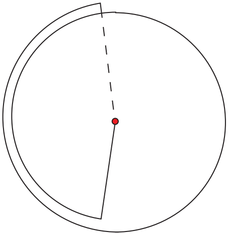
The formal solution is $\hat{f}(x) = \sum_{n=0}^{\infty}(-1)^n n!\, x^{n+1}$. Hence

$$\mathscr{B}(\hat{f})(\zeta) = \sum_{n=0}^{\infty}(-1)^n\zeta^n = \frac{1}{1+\zeta}$$

and

$$\mathscr{L}\left(\mathscr{B}(\hat{f})\right)(x) = \int_d \frac{e^{-\frac{\zeta}{x}}}{1+\zeta}\,d\zeta$$

which is a solution of the differential equation on the domain



Comparison with the solution obtained by variation of the constant

The solutions of the linear system are given by

$$f(x) = e^{\frac{1}{x}} \left(\int_0^x \frac{1}{\xi} e^{-\frac{1}{\xi}} \, d\xi + C \right)$$

For $x > 0$, the solution which is asymptotic to 0 corresponds to $C = 0$.

The change of variable $\frac{1}{x} - \frac{1}{\xi} = -\frac{\zeta}{x}$ provides the solution given by Borel's summation method

$$f(x) = \int_0^\infty \frac{e^{-\frac{\zeta}{x}}}{1+\zeta} d\zeta$$

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- $y = \hat{f}(x)$ is a formal solution of the differential equation

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- and if $\hat{f}(x)$ is absolutely Borel-summable with sum $f(x)$,

then the function $y = f(x)$ is solution of the differential equation (*).

Approximation of the function by a partial sum

Let us show that, if we approximate the solution

$$f(x) = \int_0^\infty \frac{e^{-\frac{\zeta}{x}}}{1+\zeta}\, d\zeta$$

by a partial sum

$$S_n(x) = \sum_{k=0}^{n} (-1)^k k!\, x^{k+1}$$

then, for $x > 0$, the error is smaller than the first neglected term

$$|f(x) - S_n(x)| \leq (n+1)!\, x^{n+2}$$

Details

$$
\begin{aligned}
S_n(x) &= \sum_{k=0}^{n} (-1)^k k!\, x^{k+1} \\
&= \sum_{k=0}^{n} (-1)^k x^{k+1} \int_0^\infty e^{-y} y^k \, dy \\
&= \int_0^\infty \sum_{k=0}^{n} (-1)^k x^{k+1} e^{-y} y^k \, dy \qquad y = \frac{\zeta}{x} \\
&= \int_0^\infty \sum_{k=0}^{n} (-1)^k e^{-\frac{\zeta}{x}} \zeta^k \, d\zeta
\end{aligned}
$$

Details

Then

$$|f(x)-S_n(x)| \leq \int_0^\infty e^{-\frac{\zeta}{x}} \underbrace{\left|\frac{1}{1+\zeta}-\sum_{k=0}^{n}(-1)^k\zeta^k\right|}_{\left|\frac{(-1)^{n+1}\zeta^{n+1}}{1+\zeta}\right|} d\zeta$$

Details (end)

$$
\begin{aligned}
|f(x) - S_n(x)| &\le \int_0^\infty e^{-\frac{\zeta}{x}} \left| \frac{(-1)^{n+1}\zeta^{n+1}}{1+\zeta} \right| d\zeta \\
&\le \int_0^\infty e^{-\frac{\zeta}{x}} \zeta^{n+1} d\zeta \qquad \zeta = xy \\
&= \int_0^\infty e^{-y} x^{n+2} y^{n+1}\, dy \\
&= (n+1)!\, x^{n+2}
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Quality of the approximation

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Of the order of

$$(n+1)!\,x^{n+2} \sim (n+1)\left(\sqrt{2\pi}\,e^{-(n+1)}(n+1)^{n+\frac{1}{2}}\right)\frac{1}{(n+1)^{n+2}}$$

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$$\sim \sqrt{2\pi}\sqrt{x}\, e^{-\frac{1}{x}}$$

The approximation is exponentially precise!

Quality of the approximation

This example is far from an exception. Poincaré would have said that this series is “convergent for the astronomers”:

“On the contrary, the astronomers use to say that a series converges when, for instance, the first 20 terms decrease very rapidly, even if the remaining terms would grow forever.”

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The future

What does Abel mean when he says that divergent series appear in *the most essential part of mathematics*?

There are indeed in mathematics many situations where divergence is the rule and convergence the exception.

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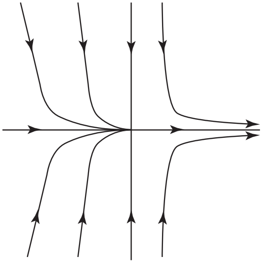
Why?

The center manifold of a saddle node

The center manifold of a saddle-node of a 2-dimensional analytic vector field is almost never analytic.

$$\dot{x} = x^2$$
$$\dot{y} = -y + x$$



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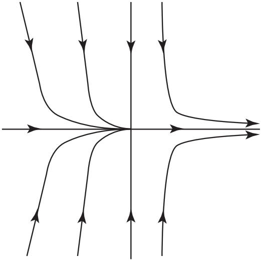
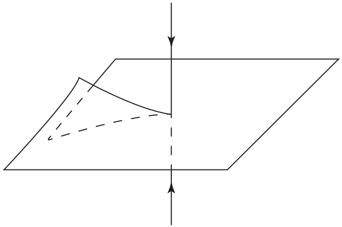


The center manifold is given by Euler's equation

$$x^2 y' + y - x = 0$$

To understand we complexify $x, y \in \mathbb{C}$

$$\dot{x} = x^2$$
$$\dot{y} = -y + x$$



Why is the center manifold generically ramified at a saddle-node?

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Because there is a hidden singularity in one direction. . .

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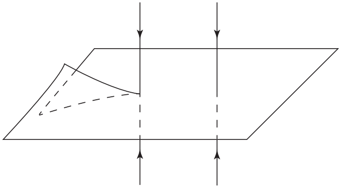
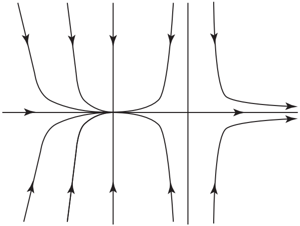
Because there is a hidden singularity in one direction...

To understand we unfold

$$\dot{x} = x^2 - \epsilon$$
$$\dot{y} = -y + x$$

Indeed, the saddle has an analytic invariant manifold. At the limit the node becomes the hidden singularity.





But why should we have ramification at the node generically?

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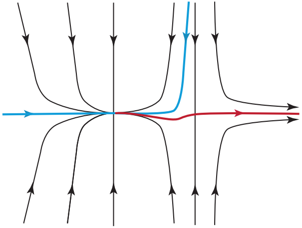
Normal form at a node if $2\sqrt{\epsilon} \notin -1/\mathbb{N}$

$$\dot{x} = -2\sqrt{\epsilon}x$$
$$\dot{y} = -y$$

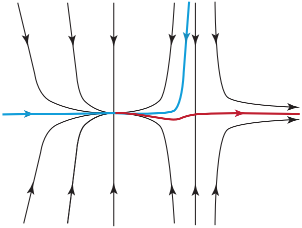


All solutions are of the form

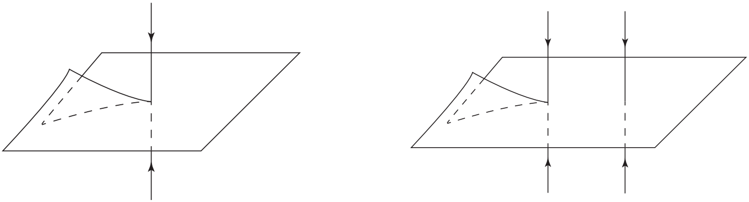
$y = Cx^{-\frac{1}{2\sqrt{\epsilon}}}$. They are all ramified except the one for $C = 0$

Non resonant case

The generic situation is that the analytic invariant manifolds of the saddle and the node do not match.



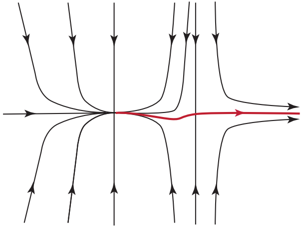
This is necessarily the case for small ϵ when unfolding a saddle-node with non analytic center manifold.



Case of a resonant node

Normal form at a node if $2\sqrt{\epsilon} \in -1/\mathbb{N}$

$$\dot{x} = -\frac{1}{n}x$$
$$\dot{y} = -y + ax^n$$



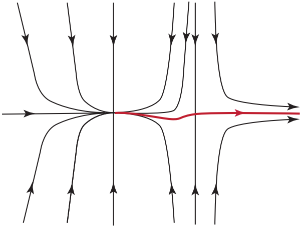
The solutions are the form

$$y = Cx^n - ax^n \log x.$$

They are all ramified if $a \neq 0$, and none is ramified if $a = 0$.

Parametric resurgence phenomenon in the resonant case

When unfolding a saddle-node with non analytic center manifold then, for small ϵ, the node is necessarily non linearizable when resonant.



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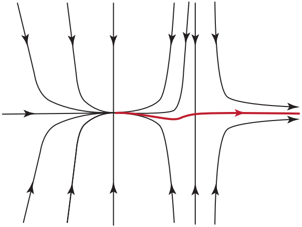

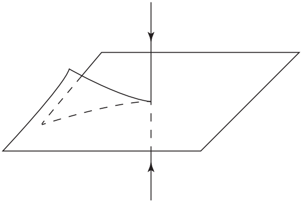

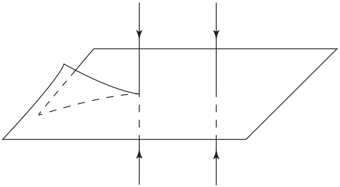

Conclusion

We have understood why divergence is the rule, and convergence, the exception.

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The phenomenon described above is quite generic and geometric explanations in this spirit are valid in many examples.

In dynamical systems

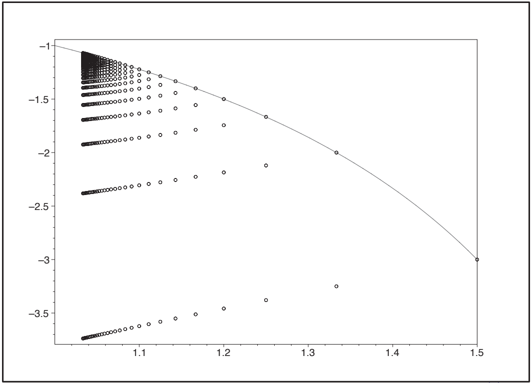
- Borel-summability (also called 1-*summability*) occurs when 2 special *objects* (singular points, limit cycles, etc.) coallesce.
- k-*summability* or *multisummability* occurs when $k+1$ special objects coallesce.
- Divergence could occur through *small divisors*: an explanation was proposed by Yoccoz and further studied by Perez-Marco.

An example studied with Colin Christopher

The saddle point in

$$\dot{x} = x(1 - x + y)$$
$$\dot{y} = y(-\lambda + x + dy)$$

is linearizable for all values of d, except when $\lambda = 1 + \frac{1}{n}$, in which case it is integrable only for the values of d given by the dots.



The end