Hints for exercises in chapter 6

Exercise 6.1.2. Study where lines of rational slope, going through the point $(2, 1)$, hit the curve again.

Exercise 6.1.5. Write down an equation that identifies when three given squares are in arithmetic progression.

Exercise 6.3.1(a). By (6.1.1) the area is $g^2rs(r^2-s^2)$ where $r > s \geq 1$ and $(r, s) = 1$. If this is a square, then each of r, s, and $r^2 - s^2$ must be squares; call them x^2, y^2, and z^2, respectively, so that $x^4 - y^4 = z^2$, which contradicts Theorem 6.2.

(c) Consider a right-angled triangle with sides $x^2, 2y^2, z$.

Exercise 6.5.3. Here b is the hypotenuse, and c is the area. Further hint: We need $b^2 - 4c$ and $b^2 + 4c$ to be integer squares, say, u^2 and v^2, so that $4c = b^2 - u^2 = v^2 - b^2$. Therefore $2b^2 = u^2 + v^2$, so u, v have the same parity and therefore $(\frac{u+v}{2})^2 + (\frac{u-v}{2})^2 = b^2$. This is our Pythagorean triangle, which has area $\frac{1}{2} \cdot \frac{u+v}{2} \cdot \frac{v-u}{2} = \frac{v^2-b^2+b^2-u^2}{8} = c$.

Exercise 6.5.6. Let $\alpha = p/q$ with $(p, q) = 1$ so that $\alpha = (a\alpha + b)/\alpha = (ap + bq)/p$. Now $(p, q) = 1$ so comparing denominators we must have $q = 1$, and p divides $ap + bq$, so that p divides bq, and therefore b.

Exercise 6.5.7. By (6.1.1) the perimeter of such a triangle has length $2grs + g(r^2 - s^2) + g(r^2 + s^2) = 2gr(r + s)$ where $r > s > 0$. Therefore n has divisors r and $r + s$, where $r < r + s < 2r$. On the other hand if n has divisors d_1, d_2 for which $d_1 < d_2 < 2d_1$, then we may assume they are coprime, by dividing through by any common factor. Therefore d_1d_2 divides n and so we can let $r = d_1$, $s = d_2 - d_1$, and $g = n/d_1d_2$.

Exercise 6.5.9. Prove that if $n \geq 13$, then $(n + 1)^2 + 128 < 2n^2$. Then proceed by induction on n for $m \in [n^2 + 129, 2n^2)$.

Exercise 6.5.10. What values can cubes take mod 9?

Exercise 6.8.1. By simple geometry things must look like the following diagram.

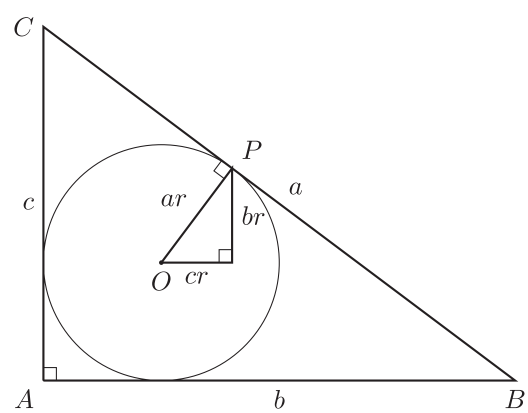

Figure. A circle inscribed inside a right-angled triangle.

If $A = (0,0)$, then BC is the line $by + cx = bc$. The point $P = (ar + cr, br + ar)$ lies on this line, and so $bc = r(b(b+a) + c(a+c)) = r(b^2 + ab + c^2 + ac) = ar(a+b+c)$. Therefore the radius of the circle is

$$ar = \frac{bc}{a+b+c} = \frac{bc(b+c-a)}{(b+c+a)(b+c-a)} = \frac{bc(b+c-a)}{b^2+c^2-a^2+2bc} = \frac{b+c-a}{2}.$$

Exercise 6.10.1. First prove that $(n+i)(n+j) \neq (n+I)(n+J)$ for $k \geq I, J, i, j \geq 1$ unless $\{I,J\} = \{i,j\}$: Suppose that $(n+i)(n+j) = (n+I)(n+J)$ so that $(i+j-I-J)n = IJ - ij$. If $I = J = k$ or $I = J = 1$, then evidently $i = j = I = J$. Therefore $k(k-1) \geq IJ \geq 2$, and similarly ij, so that $IJ - ij \equiv 0 \pmod{n}$ and $n > k^2 - k > |IJ - ij|$. Therefore $IJ = ij$ and so $I + J = i + j$, which implies that $\{I,J\} = \{i,j\}$. Now if $a_I a_J = a_i a_j$ with $\{I,J\} \neq \{i,j\}$, we may suppose that $(n+I)(n+J) > (n+i)(n+j)$. Therefore

$$\begin{aligned}(n+I)(n+J) - (n+i)(n+j) &= a_i a_j((m_I m_J)^\ell - (m_i m_j)^\ell) \\ &\geq a_i a_j((m_i m_j + 1)^\ell - (m_i m_j)^\ell) > \ell a_i m_i^{\ell-1} a_j m_j^{\ell-1} \\ &> \ell((n+i)(n+j))^{1-1/\ell} > 3n^{4/3} > 3n(k-1)\end{aligned}$$

as $n > k^\ell - k > (k-1)^3$. Therefore

$$3n(k-1) < (n+I)(n+J) - (n+i)(n+j) = (I+J-i-j)n + (IJ-ij) \leq (k-1)(2n+k+1),$$

which is a contradiction.