Hints for exercises in chapter 4

Exercise 4.0.1. One can proceed by induction on the number of distinct prime factors of n, using the definition of multiplicative.

Exercise 4.1.3. Pair m with $n-m$, and then m with n/m.

Exercise 4.1.5. If the prime factors of n are $p_1 < p_2 < \cdots < p_k$, then $p_j \geq k+j$ and so $\frac{\phi(n)}{n} = \prod_{j=1}^{k} \frac{p_j - 1}{p_j} \geq \prod_{j=1}^{k} \frac{k+j-1}{k+j} = \frac{k}{2k} = \frac{1}{2}$.

Exercise 4.2.2. Let $\ell = (d, a)$ so that $\ell | a$ and therefore $d/\ell | (a/\ell)b$ with $(d/\ell, a/\ell) = 1$ and therefore $m = d/\ell | b$.

Exercise 4.2.3(b). What is the power of 2 in $\sigma(n)$?

Exercise 4.2.4. Give a general lower bound on $\sigma(n)$.

Exercise 4.2.5(a). If $p^e \| n$, then $1 + \frac{1}{p} \leq \sigma(p^e)/p^e < 1 + \frac{1}{p} + \frac{1}{p^2} + \cdots = \frac{p}{p-1}$.

(b) If n is a perfect number, then $\sigma(n)/n = 2$, and if it is odd with ≤ 2 prime factors, then $\prod_{p|n} \frac{p}{p-1} \leq \frac{3}{2} \cdot \frac{5}{4}$ which is < 2, contradicting (a).

Exercise 4.3.7(a). Use exercise 3.9.15(a).

Exercise 4.3.11(a). Prove this when a and b are both powers of a fixed prime and then use multiplicativity.

Exercise 4.3.12. In both parts write, for each $d|n$, the integers $m = an/d$ with $(a, d) = 1$. Use exercise 4.1.3.

Exercise 4.3.13(a). You could use the second part of exercise 4.1.3.

Exercise 4.3.15(b). Use multiplicativity. (e) Use exercise 4.2.5.

Exercise 4.5.1(a). Use the binomial theorem. (b) Let $m = \prod_{p|n} p$ and $x = -1$ in (a).

Exercise 4.5.2. Expand the right-hand side.

Exercise 4.6.2. Let $r = (a, m)$ and then $s = a/r$ and $t = m/r$ which therefore must be coprime. Now $a = rs$ divides $mn = rtn$, so that s divides tn and therefore s divides n as $(s, t) = 1$. Let $u = n/s$ and we finally deduce $b = mn/a = tu$.

Exercise 4.8.2. Use the expansion $\phi(n) = \sum_{d|n} \mu(n/d)d$ from the proof of Theorem 4.1 in section 4.4, and a similar expression for σ.

Exercise 4.9.1(b). How large is the set $\{n \geq 1 : \ |f(n)| > n^\sigma\}$?

Exercise 4.11.1. For $n = 1, 2, \ldots$ determine the coefficient of t^n in $a(t)b(t) = 1$ as a polynomial in the a_i and b_j, and then find where a given b_m first appears.

Exercise 4.14.1(c). Use that $\{t\} = t - [t]$, and cut the integral up into intervals $[n, n+1)$, taking the integral up to integer N and then letting $N \to \infty$.

Exercise 4.15.1(a). Write $t = n + u$, subtract $\log n$, and then integrate the first few terms of the resulting Taylor series.

Exercise 4.16.1(b). Use the Fundamental Theorem of Algebra.

Exercise 4.16.4. Use the Möbius inversion formula from section 4.6.