Hints for exercises in chapter 0

Exercise 0.1.1(b). The key observation is that if $\alpha = \frac{1+\sqrt{5}}{2}$ or $\frac{1-\sqrt{5}}{2}$, then $\alpha^2 = \alpha+1$ and so, multiplying through by α^{n-2}, we have $\alpha^n = \alpha^{n-1} + \alpha^{n-2}$ for all $n \geq 2$.

Exercise 0.1.3(b). Multiplying through by ϕ we have $\phi^{n+1} = F_n\phi^2 + F_{n-1}\phi$. Now use (a).

Exercise 0.1.5(b). Determine a and b in terms of α and then c and d in terms of α, x_0, and x_1.

Exercise 0.2.1(a). Note that $N^2 + (2N+1) = (N+1)^2$.

Exercise 0.3.1. In both parts use induction on n.

Exercise 0.4.2. Use (0.1.1) to establish that $|F_n - \phi^n/\sqrt{5}| < \frac{1}{2}$ for all $n \geq 0$.

Exercise 0.4.7. If the first character in a string in A_n is a 0, what must the subsequent string look like? What if the string begins with a 1?

Exercise 0.4.8. Use Gauss's trick to show that $\sum_{a<n\leq b} n = \binom{b+1}{2} - \binom{a+1}{2} = \frac{(b-a)(b+a+1)}{2}$, a product of two integers of opposite parity, both > 1. Show that if N is not a power of 2 (so that it has an odd divisor $m > 1$), then it is a product of two integers of opposite parity, both > 1. Determine a and b in terms of N and m.

Exercise 0.4.10(a). Verify this for $k = 1$ and 2, and then for larger k by induction.
(b) Select k and m as functions of n.

Exercise 0.4.16. By (0.1.1), $\sqrt{5}F_n = \phi^n - \overline{\phi}^n$, and so $(\sqrt{5}F_n)^k = \sum_{j=0}^{k}\binom{k}{j}(-1)^j\rho_j^n$ where $\rho_j := \overline{\phi}^j\phi^{k-j}$. Let $x^{k+1} - \sum_{i=0}^{k} c_i x^i = \prod_{j=0}^{k}(x-\rho_j)$. Therefore

$$\sum_{i=0}^{k} c_i(\sqrt{5}F_{n+i})^k = \sum_{j=0}^{k}\binom{k}{j}(-1)^j\rho_j^n\cdot\sum_{i=0}^{k} c_i\rho_j^i = \sum_{j=0}^{k}\binom{k}{j}(-1)^j\rho_j^n\cdot\rho_j^{k+1} = (\sqrt{5}F_{n+k})^k.$$

The result follows after dividing through by $(\sqrt{5})^k$.

Exercise 0.6.1(a). Prove this for $k = 0$, and then by induction on k, using differential calculus.

Exercise 0.18.3. Substitute the value of y given by the line, into the equation of the circle.

Exercise 0.18.4. Subtract the equations for the two circles, and use exercise 0.18.3.