Chapter 4

Multiplicative functions

Exercise 4.0.1. Show that if f is multiplicative and $n = \prod_{p \text{ prime}} p^{e_p}$, then

$$f(n) = \prod_{p \text{ prime}} f(p^{e_p}).$$

Deduce that if f is totally multiplicative, then $f(n) = \prod_p f(p)^{e_p}$.

Exercise 4.0.2. Prove that if f is a multiplicative function, then either $f(n) = 0$ for all $n \geq 1$ or $f(1) = 1$.

Exercise 4.0.3. Prove that if f and g are multiplicative functions, then so is h, where $h(n) = f(n)g(n)$ for all $n \geq 1$.

Exercise 4.0.4. Prove that if f is completely multiplicative and $d|n$, then $f(d)$ divides $f(n)$.

Exercise 4.0.5. Prove that if f is multiplicative and a and b are any two positive integers, then

$$f((a,b))f([a,b]) = f(a)f(b).$$

4.1. Euler's ϕ-function

Exercise 4.1.1. Prove that if $d|n$, then $\phi(d)$ divides $\phi(n)$.

Exercise 4.1.2. Prove that if n is odd and $\phi(n) \equiv 2 \pmod 4$, then n has exactly one prime factor (perhaps repeated several times).

Exercise 4.1.3. Prove that $\sum_{1 \leq m \leq n,\ (m,n)=1} m = n\phi(n)/2$ and $\prod_{d|n} d = n^{\tau(n)/2}$.

Exercise 4.1.4. (a) Prove that $\phi(n^2) = n\phi(n)$.
(b) Prove that if $\phi(n)|n-1$, then n is squarefree.
(c) Find all integers n for which $\phi(n)$ is odd.

Exercise 4.1.5.† Suppose that n has exactly k prime factors, each of which is $> k$. Prove that $\phi(n) \geq n/2$.

4.2. Perfect numbers. *"The whole is equal to the sum of its parts."*

Exercise 4.2.1. Show that $\sigma(n) = \sum_{d|n} n/d$, and so deduce that n is perfect if and only if $\sum_{d|n} \frac{1}{d} = 2$.

Exercise 4.2.2. (a) Prove that each divisor d of ab can be written as ℓm where $\ell|a$ and $m|b$.
(b) Show that if $(a, b) = 1$, then there is a unique such pair ℓ, m for each divisor d.

Exercise 4.2.3. (a) Prove that if p is odd and k is odd, then $\sigma(p^k)$ is even.
(b)† Deduce that if n is an odd perfect number, then $n = p^\ell m^2$ where p is a prime that does not divide the integer $m \geq 1$ and $p \equiv \ell \equiv 1 \pmod 4$.

Exercise 4.2.4. Fix integer $m > 1$. Show that there are only finitely many integers n for which $\sigma(n) = m$.

Exercise 4.2.5.† (a) Prove that for all integers $n > 1$ we have the inequalities

$$\prod_{p|n} \frac{p+1}{p} \leq \frac{\sigma(n)}{n} < \prod_{p|n} \frac{p}{p-1}.$$

(b) We have seen that every even perfect number has exactly two distinct prime factors. Prove that every odd perfect number has at least three distinct prime factors.

Additional exercises

Exercise 4.3.1. Suppose that $f(n) = 0$ if n is even, $f(n) = 1$ if $n \equiv 1 \pmod 4$, and $f(n) = -1$ if $n \equiv -1 \pmod 4$. Prove that $f(.)$ is a multiplicative function.

Exercise 4.3.2.† Suppose that $r(.)$ is a multiplicative function taking values in $\mathbb{C}$. Let $f(n) = 1$ if $r(n) \neq 0$, and $f(n) = 0$ if $r(n) = 0$. Prove that $f(.)$ is also a multiplicative function.

Exercise 4.3.3.† Suppose that f is a multiplicative function, such that the value of $f(n)$ depends only on the value of $n \pmod 3$. What are the possibilities for f?

Exercise 4.3.4.‡ Suppose that f is a multiplicative function, such that the value of $f(n)$ depends only on the value of $n \pmod 8$. What are the possibilities for f?

Exercise 4.3.5. How many of the fractions a/n with $1 \leq a \leq n-1$ are reduced?

Exercise 4.3.6.† (a) Find all integers n for which $\phi(2n) = \phi(n)$.
(b) Find all integers n for which $\phi(3n) = \phi(2n)$.
(c) Can you find other classes of m for which Carmichael's conjecture is true?

Carmichael's conjecture is still an open problem but it is known that if it is false, then the smallest counterexample is $> 10^{10^{10}}$.

Exercise 4.3.7.† (a) Given a polynomial $f(x) \in \mathbb{Z}[x]$ let $N_f(m)$ denote the number of $a \pmod m$ for which $f(a) \equiv 0 \pmod m$. Show that $N_f(m)$ is a multiplicative function.
(b) Be explicit about $N_f(m)$ when $f(x) = x^2 - 1$. (You can use section 3.8.)

Exercise 4.3.8.‡ Given a polynomial $f(x) \in \mathbb{Z}[x]$ let $R_f(m)$ denote the number of $b \pmod m$ for which there exists $a \pmod m$ with $f(a) \equiv b \pmod m$. Show that $R_f(m)$ is a multiplicative function. Can you be more explicit about $R_f(m)$ for $f(x) = x^2$, the example of exercise 2.5.6?

Exercise 4.3.9. Let $\tau(n)$ denote the number of divisors of n (as in section 3.3), and let $\omega(n)$ and $\Omega(n)$ be the number of prime divisors of n not counting and counting repeated prime factors, respectively. Therefore $\tau(12) = 6, \omega(12) = 2$, and $\Omega(12) = 3$. Prove that

$$2^{\omega(n)} \leq \tau(n) \leq 2^{\Omega(n)} \text{ for all integers } n \geq 1.$$

Exercise 4.3.10. Let $\sigma_k(n) = \sum_{d|n} d^k$. Prove that $\sigma_k(n)$ is multiplicative.

Exercise 4.3.11. (a) Prove that $\tau(ab) \leq \tau(a)\tau(b)$ for all positive integers a and b, with equality if and only if $(a, b) = 1$,
(b) Prove that $\sigma_k(ab) \leq \sigma_k(a)\sigma_k(b)$ for all positive integers a, b, and k.
(c) Prove that $\sigma_{k+\ell}(n) \leq \sigma_k(n)\sigma_\ell(n)$ for all positive integers k, ℓ, and n.

Exercise 4.3.12. Give closed formulas for (a)† $\sum_{m=1}^{n} \gcd(m,n)$ and (b)‡ $\sum_{m=1}^{n} \operatorname{lcm}(m,n)$ in terms of the prime power factors of n.

Exercise 4.3.13. n is *multiplicatively perfect* if it equals the product of its proper divisors.
(a) Show that n is multiplicatively perfect if and only if $\tau(n) = 4$.
(b) Classify exactly which integers n satisfy this.

The integers m and n are *amicable* if the sum of the proper divisors of m equals n and the sum of the proper divisors of n equals m. For example, 220 and 284 are amicable, as are 1184 and 1210.

Exercise 4.3.14. (a) Show that m and n are amicable if and only if $\sigma(m) = \sigma(n) = m + n$.
(b) Verify Thâbit ibn Qurrah's 9th-century claim that if $p = 3 \times 2^{n-1} - 1,\ q = 3 \times 2^n - 1$, and $r = 9 \times 2^{2n-1} - 1$ are each odd primes, then $2^n pq$ and $2^n r$ are amicable.
(c) Find an example (other than the two given above) using the construction in (b).

Exercise 4.3.15. (a) Prove that every prime number is deficient.
(b) Prove that every multiple of 6 is abundant.
(c) How do these concepts relate to the value of $\sigma(n)/n$?
(d) Prove that every multiple of an abundant number is abundant.
(e)† Prove that if n is the product of k distinct primes that are each $> k$, then n is deficient.
(f) Prove that every divisor of a deficient number is deficient.

Exercise 4.3.16. (a) How many different 2-deep, 120-brick, rectangular formations are there?
(b) What if there must be at least three bricks along the width and along the length?

Appendix 4A. More multiplicative functions

4.4. Summing multiplicative functions

Exercise 4.4.1. Prove that $\frac{n}{\phi(n)} = \sum_{\substack{d|n \\ d \text{ squarefree}}} \frac{1}{\phi(d)}$.

4.5. Inclusion-exclusion and the Möbius function

Exercise 4.5.1. (a) Show that if m is squarefree, then

$$(1+x)^{\omega(m)} = \sum_{d|m} x^{\omega(d)}.$$

(b) Deduce Corollary 4.5.1.

Exercise 4.5.2. Prove that for any positive integer n we have

$$\sum_{d|n} \frac{\mu(d)}{d} = \prod_{\substack{p \text{ prime} \\ p|n}} \left(1 - \frac{1}{p}\right).$$

Exercise 4.5.3. Prove that $\mu(n)^2$ is the characteristic function for the squarefree integers, and deduce that $\frac{n}{\phi(n)} = \sum_{d|n} \frac{\mu(d)^2}{\phi(d)}$.

4.6. Convolutions and the Möbius inversion formula

Exercise 4.6.1. Prove that $\delta * f = f$ for all f, $\tau = 1 * 1$, and $\sigma(n) = 1 * I$.

Exercise 4.6.2. Prove that if $ab = mn$, then there exist integers r, s, t, u with $a = rs,\ b = tu,\ m = rt,\ n = su$ with $(s,t) = 1$.

Exercise 4.6.3. Prove that $(\mu * \sigma)(n) = n$ for all integers $n \geq 1$.

Exercise 4.6.4. (a) Show that $(a * f) + (b * f) = (a + b) * f$.
(b) Let $f(n) \geq 0$ for all integers $n \geq 1$. Prove that $(1 * f)(n) + (\mu * f)(n) \geq 2f(n)$ for all integers $n \geq 1$.
(c) Prove that $\sigma(n) + \phi(n) \geq 2n$ for all integers $n \geq 1$.

Exercise 4.6.5. Suppose that $g(n) = \prod_{d|n} f(d)$. Deduce that $f(n) = \prod_{d|n} g(d)^{\mu(n/d)}$.

4.7. The Liouville function

Exercise 4.7.1. Prove that $\Omega(n) \geq \omega(n)$ for all integers $n \geq 1$, with equality if and only if n is squarefree.

Exercise 4.7.2. Prove that $\lambda * \mu^2 = \delta$.

Exercise 4.7.3. (a) Prove that

$$\sum_{d|n} \lambda(d) = \begin{cases} 1 & \text{if } n \text{ is a square,} \\ 0 & \text{otherwise.} \end{cases}$$

(b)† By summing the formula in (a) over all positive integers $n \leq N$, deduce that for all integers $N \geq 1$ we have

$$\sum_{n \geq 1} \lambda(n) \left[\frac{N}{n} \right] = [\sqrt{N}].$$

Additional exercises

Exercise 4.8.1. Prove that $\mu(n)\mu(n+1)\mu(n+2)\mu(n+3) = 0$ for all integers $n \geq 1$.

Exercise 4.8.2. Prove that $\phi(n) + \sigma(n) = 2n$ if and only if $n = 1$ or n is a prime.

Exercise 4.8.3. (a) By summing the formula in Corollary 4.5.1 over all positive integers $n \leq N$, deduce that

$$\sum_{n \geq 1} \mu(n) \left[\frac{N}{n} \right] = 1 \quad \text{for all } N \geq 1.$$

(b)† Deduce that

$$\left| \sum_{n \leq N} \frac{\mu(n)}{n} \right| \leq 1 \quad \text{for all } N \geq 1.$$

It is a much deeper problem to prove that $\sum_{n \leq N} \mu(n)/n$ tends to a limit as $N \to \infty$.

Exercise 4.8.4. (a) Prove that if f is an arithmetic function, then

$$\sum_{n \geq 1} f(n) \frac{x^n}{1 - x^n} = \sum_{m \geq 1} (1 * f)(m) x^m,$$

without worrying about convergence.
(b) Write out explicitly the example $f = \mu$ as well as some other common multiplicative functions.

Appendix 4B. Dirichlet series and multiplicative functions

4.9. Dirichlet series

Exercise 4.9.1.† (a) Prove that if there exists a constant $c > 0$ and $\tau \in \mathbb{R}$ such that $|f(n)| < cn^\tau$ for all integers $n \geq 1$, then $F(s)$ is absolutely convergent for $s = \sigma + it$ with $\sigma > 1 + \tau$.
(b) Prove that if $F(s)$ is absolutely convergent for $\text{Re}(s) = \sigma$, then there exists a constant $c > 0$ such that $|f(n)| < cn^\sigma$ for all integers $n \geq 1$.

4.10. Multiplication of Dirichlet series

Exercise 4.10.1. By studying the Euler products or otherwise, prove the following:
(a) $\sum_{n\geq 1} n/n^s = \zeta(s-1)$;
(b) $\sum_{n\geq 1} \tau(n)/n^s = \zeta(s)^2$;
(c) $\sum_{n\geq 1} \sigma(n)/n^s = \zeta(s)\zeta(s-1)$ and $\sum_{n\geq 1} \sigma_k(n)/n^s = \zeta(s)\zeta(s-k)$ for all integers $k \geq 1$;
(d) $\sum_{n\geq 1} \mu(n)^2/n^s = \zeta(s)/\zeta(2s)$;
(e) $\sum_{n\geq 1} \phi(n)/n^s = \zeta(s-1)/\zeta(s)$;
(f) $\sum_{n\geq 1} \lambda(n)/n^s = \zeta(2s)/\zeta(s)$.

Exercise 4.10.2. (a) Describe the identity in Proposition 4.1.1 in terms of Dirichlet series.
(b) Reprove exercise 4.10.1(d) and (e) by multiplying through by denominators.
(c) Give a formula for the coefficients of $F(s)\zeta(s)$ in terms of the values of the $f(n)$.
(d) Suppose that $f(p)$ is totally multiplicative with $f(2) = 0$, $f(p) = 1$ if $p \equiv 1 \pmod 3$, and $f(p) = -1$ if $p \equiv -1 \pmod 3$. Describe the coefficient of $1/n^s$ in $F(s)\zeta(s)$ in terms of the prime power factors of n.

4.11. Other Dirichlet series of interest

Exercise 4.11.1.† (a) Suppose that $a(t) = 1 + a_1 t + a_2 t^2 + \cdots$. Prove that there exists an inverse $b(t) = 1 + b_1 t + b_2 t^2 + \cdots$ for which $a(t)b(t) = 1$.
(b) Deduce that $\left(1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + \cdots\right)$, the pth Euler factor of $F(s)$, is invertible.

Exercise 4.11.2. (a) Prove that $\Lambda_f(n) + \Lambda_g(n) = 0$ for all integers $n \geq 1$.
(b) Use the identity $-F'(s) = F(s) \cdot (-F'(s)/F(s))$ to prove that

$$f(n) \log n = \sum_{ab=n} f(a)\Lambda_f(b) \quad \text{for all integers } n \geq 1.$$

(c) Use the identity $-F'(s)/F(s) = G(s) \cdot (-F'(s))$ to prove that

$$\Lambda_f(n) = \sum_{ab=n} g(a)f(b) \log b \quad \text{for all integers } n \geq 1.$$

(d) Deduce (4.11.1).

Appendix 4C. Irreducible polynomials modulo p

4.12. Irreducible polynomials modulo p

Exercise 4.12.1. Suppose that $h(x)$ is a given polynomial mod p of degree d. Prove that there are exactly p^{m-d} monic polynomials in $\mathbb{F}_p[x]$ of degree m that are divisible by $h(x)$, provided $m \geq d$.

Exercise 4.12.2. Prove that $N_m \leq p^m/m$.

Appendix 4D. The harmonic sum and the divisor function

4.13. The average number of divisors

4.14. The harmonic sum

Exercise 4.14.1.[†] Let $x_n = 1/1 + 1/2 + 1/3 + \cdots + 1/n - \log n$ for each integer $n \geq 1$.

(a) By modifying the argument above, show that if $n > m$, then

$$0 \leq x_m - x_n \leq \log(1 + 1/m) - \log(1 + 1/n) < 1/m.$$

Therefore $(x_n)_{n\geq 1}$ is a Cauchy sequence and converges to a limit, as desired.

(b) Deduce that $0 \leq \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{N} - \log N - \gamma \leq \frac{1}{N}$.

(c)[‡] Prove that

$$\gamma = 1 - \int_1^\infty \frac{\{t\}}{t^2}\, dt.$$

Exercise 4.14.2.[†] Notice that $\log x$ is a monotone increasing function for $x \geq 1$. Assume that $N \geq M$.

(a) Justify the lower bound $\log n \geq \int_{n-1}^n \log t\, dt$ and deduce that $N!/M! \geq (N/e)^N/(M/e)^M$.

(b) Prove an analogous upper bound on $\log n$ and deduce that $N!/M! \leq (N/e)^{N+1}/(M/e)^{M+1}$.

(c) Deduce that $1/\sqrt{N} \leq N! \Big/ \left(\frac{N}{e}\right)^N \sqrt{e^2 N} \leq \sqrt{N}$.

4.15. Dirichlet's hyperbola trick

Exercise 4.15.1.[‡] Let $M_N := \log 1 + \log 2 + \cdots + \log N$ and $x_N := M_N - (N + \frac{1}{2})(\log N - 1)$.

(a) Prove that there exists a constant $c_1 > 0$ such that for all integers $n \geq 1$ we have

$$\left| \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \log t\, dt - \log n \right| \leq \frac{c_1}{n^2}.$$

(b) Deduce that if $M > N$, then $|x_M - x_N| < c_2/N$ for some constant $c_2 > 0$.

(c) Prove that $(x_N)_{N\geq 1}$ tends to a limit.

(d) Deduce that there exists a constant $c_0 > 0$ such that

$$\lim_{N\to\infty} \quad N! \Big/ \left(\frac{N}{e}\right)^N \sqrt{c_0 N} \;=\; 1.$$

Stirling showed that $c_0 = 2\pi$ (this amazingly accurate approximation to $N!$ is known as *Stirling's formula*).

Appendix 4E. Cyclotomic polynomials

4.16. Cyclotomic polynomials

Exercise 4.16.1. (a) Show that $\zeta_m^i = \zeta_m^j$ if and only if $i \equiv j \pmod m$.

(b) Prove that $x^m - 1 = (x-1)(x-\zeta_m)(x-\zeta_m^2)\cdots(x-\zeta_m^{m-1})$.

(c) Deduce that $x^m - y^m = (x-y)(x-\zeta_m y)\cdots(x-\zeta_m^{m-1}y)$.

(d) Deduce that if m is odd, then $x^m + y^m = (x+y)(x+\zeta_m y)\cdots(x+\zeta_m^{m-1}y)$.

Exercise 4.16.2. (a) Show that $\sum_{m|n} \deg \phi_m = n$ for all integers $n \geq 1$.

(b) Deduce that $\phi_m(x)$ has degree $\phi(m)$ for each $m \geq 1$.

Exercise 4.16.3. (a) Show that if m divides n, then $(\zeta_n^k)^m = 1$ if and only if n/m divides k.

(b) Deduce that if m divides n, then ζ_n^k is a primitive mth root of unity if and only if there exists an integer r, coprime with m, for which $k = (n/m)r$.

(c) Show that the set of primitive mth roots of unity is $\{\zeta_m^j : 1 \leq j \leq m \text{ and } (j,m) = 1\}$.

(d) Deduce that $\phi_m(x)$ has degree $\phi(m)$.

Exercise 4.16.4. By using the Möbius inversion formula, prove that

$$\phi_m(t) = \prod_{d|m} (t^d - 1)^{\mu(m/d)}.$$

Exercise 4.16.5. (a) Prove that $\prod_{m|n, m>1} \phi_m(1) = n$.

(b)† Deduce that if $m > 1$, then $\log \phi_m(1) = \Lambda(m)$, the von Mangoldt function of appendix 4B.