Chapter 13

The anatomy of integers

13.1. Rough estimates for the number of integers with a fixed number of prime factors

Exercise 13.1.1. Use Stirling's formula (exercise 4.15.1) to show that if m is the nearest integer to λ, then $e^{-\lambda}\lambda^m/m!$ is roughly $1/\sqrt{2\pi\lambda}$. This suggests the fact that if m is the closest integer to $\log\log x$, then there are roughly $x/\sqrt{2\pi \log\log x}$ integers up to x with exactly m prime factors.

13.2. The number of prime factors of a typical integer

Exercise 13.2.1. (a) Prove that

$$\sum_{\substack{p \text{ prime} \\ p\le x}} \left(\frac{x}{p} - \left[\frac{x}{p}\right]\right) \le \pi(x).$$

(b) Deduce that

$$\lim_{x\to\infty} \left| \frac{1}{x}\sum_{n\le x} \omega(n) - \sum_{\substack{p \text{ prime} \\ p\le x}} \frac{1}{p} \right| = 0.$$

Exercise 13.2.2. Show that if $a_1, \ldots, a_N$ have mean m, then

$$\frac{1}{N}\sum_{n\le N} (a_n - m)^2 = \frac{1}{N}\sum_{n\le N} a_n^2 - m^2.$$

Exercise 13.2.3. ("Almost all" integers n have about $\log\log n$ prime factors.) Show that (13.2.2) implies that there are $< 2x/(\log\log x)^{1/3}$ integers $n \le x$ for which $|\omega(n) - \log\log x| \ge (\log\log x)^{2/3}$. In other words, we have $|\omega(n) - \log\log x| < (\log\log x)^{2/3}$ for all but at most $< 2x/(\log\log x)^{1/3}$ integers $n \le x$. This is a famous result of Hardy and Ramanujan (we will develop their proof in section 13.4).

Exercise 13.2.4. Explain, by creating a simpler but analogous example, how it is possible that $\tau(n)$ can usually take values around $(\log n)^{\log 2}$, but averages about $\log n$. (You might think of 100 students taking an exam in which most do poorly, but one does well.)

13.3. The multiplication table problem

Exercise 13.3.1. Give a more formal version of Erdős's proof.

13.4. Hardy and Ramanujan's inequality

Exercise 13.4.1.[†] Give another proof that "almost all" integers n have about $\log\log n$ prime factors using (13.4.1).

Exercise 13.4.2.[‡] Let $k = [A \log\log x]$. Use (13.4.1) together with exercise 4.15.1(d) to give an upper bound on $\pi(x,k)$, and then use (13.4.3) to give a lower bound on $N(x,k)$. Deduce that once A satisfies $1 + A(\log A - 1) > (A-1)\log 2$, then $N(x,k) > 2\pi(x,k)$ for x sufficiently large.

Appendix 13A. Other anatomies

13.5. The anatomy of polynomials in finite fields

Exercise 13.5.1. Sketch a proof that almost all polynomials in $\mathbb{F}_p$ of degree $2d$ are *not* the product of two polynomials of degree d, as d gets large.

13.6. The anatomy of permutations

Exercise 13.6.1. (a) Prove that

$$\left(\sum_{a \le A/m} \frac{1}{a}\right)^m \le \sum_{\substack{a_1,\dots,a_m \ge 1 \\ a_1+\cdots+a_m \le A}} \frac{1}{a_1 \cdots a_{k-1}} \le \left(\sum_{a \le A} \frac{1}{a}\right)^m .$$

(b)[‡] Prove that if $m \le \frac{\log A}{(\log\log A)^2}$, then the two terms at either end of the inequalities in (a) differ by a multiplicative factor which gets arbitrarily close to 1 as A grows.

Exercise 13.6.2. Prove, by taking $m = k + \ell$, that

$$0 < \left(\sum_{k=1}^{N} \frac{1}{k}\right)^2 - \sum_{\substack{k,\ell \ge 1 \\ k+\ell \le N}} \frac{1}{k\ell} = \sum_{\substack{1 \le k,\ell \le N \\ k+\ell > N}} \frac{1}{k\ell} = 2 \sum_{m=N+1}^{2N} \frac{1}{m} \sum_{k=m-N}^{N} \frac{1}{k} = 2 \sum_{k=1}^{N} \frac{1}{k} \sum_{m=N+1}^{N+k} \frac{1}{m} < 2.$$

Appendix 13B. Dirichlet L-functions

13.7. Dirichlet series

Exercise 13.7.1. Let $\sigma = \mathrm{Re}(s) > 0$. Prove that

$$|L(s,\chi)| \le \begin{cases} \frac{\sigma}{\sigma-1} & \text{if } \sigma > 1, \\ \log q + |s| + 1 & \text{if } \sigma = 1, \\ \frac{\sigma}{\sigma-1} + q^{1-\sigma}\left(\frac{1}{1-\sigma} + \frac{|s|}{\sigma}\right) & \text{if } 0 < \sigma < 1. \end{cases}$$

Exercise 13.7.2.† (a) Prove that if $\sigma > 1$, then

$$\frac{1}{\phi(q)} \sum_{\chi \pmod q} \log L(\sigma, \chi) = \sum_{p^k \equiv 1 \pmod q} \frac{1}{kp^{k\sigma}}.$$

(b) Deduce that $\prod_{\chi \pmod q} L(s, \chi)$ is non-zero at $s = 1$.
(c) Prove that if $L(1, \chi) = 0$, then $L(1, \overline{\chi}) = 0$.
(d)‡ Deduce that if $L(1, \chi) = 0$, then χ is real.

Exercise 13.7.3.† (a) Show that for any integer $a, 1 \leq a \leq 9$, there are $\sim x/9$ integers $\leq x$ whose leading digit is a, where $x = 10^n$ and integer $n \to \infty$.
(b) Show that there are $\sim 5x/9$ integers $\leq x$ whose leading digit is 1, where $x = 2 \cdot 10^n$ and integer $n \to \infty$.
(c) What can we say about the density of integers whose leading digit is 1?
(d) The *logarithmic density* of a set S of positive integers up to x is given by $\frac{1}{\log x} \sum_{n \leq x,\ n \in S} \frac{1}{n}$. For any given integer $a,\ 1 \leq a \leq 9$, let S_a be the set of integers with leading coefficient a. Prove that the logarithmic density of S_a, namely

$$\lim_{x \to \infty} \frac{1}{\log x} \sum_{n \leq x,\ n \in S_a} \frac{1}{n} \text{ exists and equals } \frac{\log(1 + 1/a)}{\log 10}.$$

Exercise 13.7.4. Show that $\prod_{n=2}^{N} \frac{n^3 - 1}{n^3 + 1} = \frac{2}{3}\left(1 + \frac{1}{N(N+1)}\right)$.

Exercise 13.7.5.† (a) Use Theorem 5.3 to establish that there exist constants $0 < c_1 < c_2$ such that if $x \geq 2$, then

$$c_1 < \sum_{x < p \leq 3x} \frac{\log p}{p} < c_2.$$

(b) Deduce that there exist constants $0 < c_3 < c_4$ such that if $x \geq 6$, then

$$c_3 \log x < \sum_{p \leq x} \frac{\log p}{p} < c_4 \log x.$$

(c) In section 13.1 we claimed that $\sum_{\substack{p \leq \sqrt{x} \\ p \text{ prime}}} \frac{x/p}{\log x/p}$ is well-approximated by $\frac{x}{\log x} \sum_{\substack{p \leq \sqrt{x} \\ p \text{ prime}}} \frac{1}{p}$. Show that there exists a constant $c_5 > 0$ such that the difference between these two expressions is $\leq c_5 x$.
(d) Prove (13.4.2).

Exercise 13.7.6. (a) Show that every integer n can be written as mr where m is powerful, r is squarefree, and $(m, r) = 1$; and deduce that $\Omega(n) - \omega(n) = \Omega(m) - \omega(m)$.
(b) Prove that there are $\leq x/m$ integers $n \leq x$ of the form mr as in (a).
(c) Prove that if $\Omega(m) - \omega(m) \geq k$, then $m \geq 2^{k+1}$. Deduce that

$$\frac{1}{x} \#\{n \leq x : \Omega(n) - \omega(n) \geq k\} \leq \sum_{\substack{m \text{ powerful} \\ m \geq 2^{k+1}}} \frac{1}{m}.$$

(d) Prove that every powerful number m can be written as $a^2 b^3$ for some integers a and b.
(e) Deduce that if $a^2 b^3 \geq 2^{k+1}$, then $a \geq 2^{k/4}$ or $b \geq 2^{k/6}$, and therefore that

$$\sum_{\substack{a, b \geq 1 \\ a^2 b^3 \geq 2^{k+1}}} \frac{1}{a^2 b^3} \leq \sum_{a \geq 2^{k/4}} \frac{1}{a^2} \sum_{b \geq 1} \frac{1}{b^3} + \sum_{a \geq 1} \frac{1}{a^2} \sum_{b \geq 2^{k/6}} \frac{1}{b^3} < \frac{5}{2^{k/4}}.$$

(f) Deduce that there are $< x/2^\ell$ integers $n \leq x$ for which $\Omega(n) - \omega(n) \geq 4\ell + 3$.

Exercise 13.7.7. (a) Use (13.4.1) to show that

$$\#\{a, b \leq x : \omega(a) + \omega(b) = k\} \leq c_0^2 \frac{x^2}{(\log x)^2} \frac{(2(\log\log x + c_1))^{k-2}}{(k-2)!}.$$

(b) Write $K := [\frac{\log\log x}{\log 2}]$ and let $\delta = 1 - \frac{1+\log\log 2}{\log 2} = .086071\ldots$. Use Stirling's formula to prove that there exists a constant c_2 such that

$$\#\{a, b \leq x : \omega(a) + \omega(b) \leq K\} + \#\{n \leq x^2 : \omega(n) > K\} \leq c_2 \frac{x^2}{(\log x)^\delta}.$$

(c) Use this result to more or less justify the claim that there are $\leq c_2 N^2/(\log N)^\delta$ distinct integers in the N-by-N multiplication table.