The anatomy of integers

13.1. Rough estimates for the number of integers with a fixed number of prime factors

Exercise 13.1.1. Use Stirling’s formula (exercise 4.15.1) to show that if $m$ is the nearest integer to $\lambda$, then $e^{-\lambda} \lambda^m / m!$ is roughly $1/\sqrt{2\pi\lambda}$. This suggests the fact that if $m$ is the closest integer to $\log \log x$, then there are roughly $x/\sqrt{2\pi \log \log x}$ integers up to $x$ with exactly $m$ prime factors.

13.2. The number of prime factors of a typical integer

Exercise 13.2.1. (a) Prove that 
\[
\sum_{p \text{ prime} \atop p \leq x} \left( \frac{x}{p} - \left\lfloor \frac{x}{p} \right\rfloor \right) \leq \pi(x).
\]

(b) Deduce that 
\[
\lim_{x \to \infty} \left| \frac{1}{x} \sum_{n \leq x} \omega(n) - \sum_{p \text{ prime}} \frac{1}{p} \right| = 0.
\]

Exercise 13.2.2. Show that if $a_1, \ldots, a_N$ have mean $m$, then 
\[
\frac{1}{N} \sum_{n \leq N} (a_n - m)^2 = \frac{1}{N} \sum_{n \leq N} a_n^2 - m^2.
\]

Exercise 13.2.3. (“Almost all” integers $n$ have about $\log \log n$ prime factors.) Show that $[3.2.2]$ implies that there are $< 2x/(\log \log x)^{1/3}$ integers $n \leq x$ for which $|\omega(n) - \log \log x| = (\log \log x)^{2/3}$. In other words, we have $|\omega(n) - \log \log x| < (\log \log x)^{2/3}$ for all but at most $< 2x/(\log \log x)^{1/3}$ integers $n \leq x$. This is a famous result of Hardy and Ramanujan (we will develop their proof in section $[3.4]$).

Exercise 13.2.4. Explain, by creating a simpler but analogous example, how it is possible that $\tau(n)$ can usually take values around $(\log n)^{\log 2}$, but averages about $\log n$. (You might think of 100 students taking an exam in which most do poorly, but one does well.)
13.3. The multiplication table problem

Exercise 13.3.1. Give a more formal version of Erdős’s proof.

13.4. Hardy and Ramanujan’s inequality

Exercise 13.4.1.† Give another proof that “almost all” integers $n$ have about $\log \log n$ prime factors using (13.4.1).

Exercise 13.4.2.‡ Let $k = \lfloor A \log \log x \rfloor$. Use (13.4.1) together with exercise 13.4.1(d) to give an upper bound on $\pi(x, k)$, and then use (13.4.3) to give a lower bound on $N(x, k)$. Deduce that once $A$ satisfies $1 + A(\log A - 1) > (A - 1) \log 2$, then $N(x, k) > 2\pi(x, k)$ for $x$ sufficiently large.

Appendix 13A. Other anatomies

13.5. The anatomy of polynomials in finite fields

Exercise 13.5.1. Sketch a proof that almost all polynomials in $\mathbb{F}_p$ of degree $2d$ are not the product of two polynomials of degree $d$, as $d$ gets large.

13.6. The anatomy of permutations

Exercise 13.6.1. (a) Prove that
\[
\left( \sum_{a \leq A/m} \frac{1}{a} \right)^m \leq \sum_{a_1 \cdots a_{k-1} \leq A} \frac{1}{a_1 \cdots a_{k-1}} \leq \left( \sum_{a \leq A} \frac{1}{a} \right)^m.
\]
(b)† Prove that if $m \leq \frac{\log A}{(\log \log A)^2}$, then the two terms at either end of the inequalities in (a) differ by a multiplicative factor which gets arbitrarily close to 1 as $A$ grows.

Exercise 13.6.2. Prove, by taking $m = k + \ell$, that
\[
0 < \left( \sum_{k=1}^{N} \frac{1}{k} \right)^2 - \sum_{k, \ell \geq 1} \frac{1}{k} \leq 2 \sum_{m=1}^{N} \frac{1}{m} \sum_{k=1}^{N} \frac{1}{k} = 2 \sum_{k=1}^{N} \frac{1}{k} \sum_{m=1}^{N} \frac{1}{m} < 2.
\]

Appendix 13B. Dirichlet $L$-functions

13.7. Dirichlet series

Exercise 13.7.1. Let $\sigma = \text{Re}(s) > 0$. Prove that
\[
|L(s, \chi)| \leq \begin{cases} 
\frac{\sigma}{\pi} & \text{if } \sigma > 1, \\
\log q + |s| + 1 & \text{if } \sigma = 1, \\
\frac{\sigma}{\pi} + q^{1-\sigma} \left( \frac{1}{1-\sigma} + |s| \right) & \text{if } 0 < \sigma < 1.
\end{cases}
\]
Exercise 13.7.2.† (a) Prove that if $\sigma > 1$, then
\[
\frac{1}{\phi(q)} \sum_{\chi \pmod{q}} \log L(\sigma, \chi) = \sum_{p^{k} \equiv 1 \pmod{q}} \frac{1}{kp^{\sigma}}.
\]
(b) Deduce that $\prod_{q} L(\sigma, \chi)$ is non-zero at $s = 1$.
(c) Prove that if $L(1, \chi) = 0$, then $L(1, \overline{\chi}) = 0$.
(d)† Deduce that if $L(1, \chi) = 0$, then $\chi$ is real.

Exercise 13.7.3.† (a) Show that there exists a constant $c$ such that if $x \rightarrow \infty$, there are $\sim \frac{x}{9}$ integers $\leq x$ whose leading digit is $a$, where $x = 10^{n}$ and integer $n \rightarrow \infty$.
(b) Show that there are $\sim \frac{5x}{9}$ integers $\leq x$ whose leading digit is 1, where $x = 2 \cdot 10^{n}$ and integer $n \rightarrow \infty$.
(c) What can we say about the density of integers whose leading digit is 1?
(d) The logarithmic density of a set $S$ of positive integers up to $x$ is given by $\frac{1}{\log x} \sum_{n \leq x, n \in S} \frac{1}{n}$. For any given integer $a$, $1 \leq a \leq 9$, let $S_{a}$ be the set of integers with leading coefficient $a$.
Prove that the logarithmic density of $S_{a}$, namely
\[
\lim_{x \rightarrow \infty} \frac{1}{\log x} \sum_{n \leq x, n \in S_{a}} \frac{1}{n}
\]
equals $\log(1 + 1/a) / \log 10$.

Exercise 13.7.4. Show that $\prod_{a=2}^{9} \frac{a^{3} - 1}{a^{3} + 1} = \frac{2}{3} \left(1 + \frac{1}{N(N+1)}\right)$.

Exercise 13.7.5.† (a) Use Theorem 5.2 to establish that there exist constants $0 < c_{1} < c_{2}$ such that if $x \geq 2$, then
\[
c_{1} < \sum_{x < p \leq 3x} \frac{\log p}{p} < c_{2}.
\]
(b) Deduce that there exist constants $0 < c_{3} < c_{4}$ such that if $x \geq 6$, then
\[
c_{3} \log x < \sum_{p \leq x} \frac{\log p}{p} < c_{4} \log x.
\]
(c) In section 13.4, we claimed that $\sum_{p \leq \sqrt{x}} \frac{\log p}{p}$ is well-approximated by $\frac{x}{\log x} \sum_{p \leq \sqrt{x}} \frac{1}{p}$.
Show that there exists a constant $c_{5} > 0$ such that the difference between these two expressions is $\leq c_{5} \cdot x$.
(d) Prove (13.4.2).

Exercise 13.7.6. (a) Show that every integer $n$ can be written as $mr$ where $m$ is powerful, $r$ is squarefree, and $(m, r) = 1$; and deduce that $\Omega(n) - \omega(n) = \Omega(m) - \omega(m)$.
(b) Prove that there are $\leq x/m$ integers $n \leq x$ of the form $mr$ as in (a).
(c) Prove that if $\Omega(m) - \omega(m) \geq k$, then $m \geq 2^{k+1}$. Deduce that
\[
\frac{1}{x} \#\{n \leq x : \Omega(n) - \omega(n) \geq k\} \leq \sum_{m \text{ powerful}} \frac{1}{m}.
\]
(d) Prove that every powerful number $m$ can be written as $a^{2}b^{3}$ for some integers $a$ and $b$.
(e) Deduce that if $a^{2}b^{3} \geq 2^{k+1}$, then $a \geq 2^{b/4}$ or $b \geq 2^{b/6}$, and therefore that
\[
\sum_{a^{2}b^{3} \geq 2^{k+1}} \frac{1}{a^{2}b^{3}} \leq \sum_{a \geq 2^{b/4}} \frac{1}{a^{2}} \sum_{b \geq 2^{b/6}} \frac{1}{b^{3}} + \sum_{a \geq 2^{b/4}} \frac{1}{a^{2}} \sum_{b \geq 2^{b/6}} \frac{1}{b^{3}} < \frac{5}{2^{b/4}}.
\]
(f) Deduce that there are $< x/2^{\ell}$ integers $n \leq x$ for which $\Omega(n) - \omega(n) \geq 4\ell + 3$.  


Exercise 13.7.7. (a) Use (13.4.1) to show that

$$\#\{a, b \leq x : \omega(a) + \omega(b) = k\} \leq c_0^2 \frac{x^2}{(\log x)^2} \frac{(2(\log \log x + c_1))^k}{(k-2)!}.$$

(b) Write $K := \left\lfloor \frac{\log \log x}{\log 2}\right\rfloor$ and let $\delta = 1 - \frac{1+\log \log 2}{\log 2} = .086071 \ldots$. Use Stirling’s formula to prove that there exists a constant $c_2$ such that

$$\#\{a, b \leq x : \omega(a) + \omega(b) \leq K\} + \#\{n \leq x^2 : \omega(n) > K\} \leq c_2 \frac{x^2}{(\log x)^2}.$$

(c) Use this result to more or less justify the claim that there are $\leq c_2 N^2/(\log N)^\delta$ distinct integers in the $N$-by-$N$ multiplication table.