Appendix 5A. Bertrand's postulate and beyond

5.9. Bertrand's postulate

In 1845 Bertrand conjectured, on the basis of calculations up to a million:

Theorem 5.6 (Bertrand's postulate)**.** *For every integer $n \geq 1$, there is a prime number between n and $2n$.*

Bertrand's postulate was proved in 1850 by Chebyshev. We will follow the 19-year-old Erdős's proof, or, as N. J. Fine put it (in the voice of Erdős):

Chebyshev said it, but I'll say it again:
There's always a prime between n and $2n$.

Exercise 5.9.1. Show that prime p does not divide $\binom{2n}{n}$ when $2n/3 < p \leq n$.

Proof of Bertrand's postulate. Let p^{e_p} be the exact power of prime p dividing $\binom{2n}{n}$. We know that

- $e_p = 1$ if $n < p \leq 2n$ by Kummer's Theorem (Theorem 3.7),
- $e_p = 0$ if $2n/3 < p \leq n$ by exercise 5.9.1,
- $e_p \leq 1$ if $\sqrt{2n} < p \leq 2n$ by Corollary 3.10.1,
- $p^{e_p} \leq 2n$ if $p \leq 2n$ by Corollary 3.10.1.

Combining these gives

$$\frac{2^{2n}}{2n} \leq \binom{2n}{n} = \prod_{p \leq 2n} p^{e_p} \leq \prod_{n<p\leq 2n} p \prod_{p \leq 2n/3} p \prod_{p \leq \sqrt{2n}} 2n$$
$$\leq \left(\prod_{n<p\leq 2n} p \right) \times 4^{2n/3-1} \times (2n)^{(\sqrt{2n}+1)/2},$$

using Lemma 5.5.1 to bound $\prod_{p \le 2n/3} p$ and the bound $\pi(\sqrt{2n}) \le \frac{1}{2}(\sqrt{2n}+1)$ (as neither 1 nor any even integer > 2 is prime). Taking logarithms we deduce that

$$\sum_{\substack{p \text{ prime} \\ n < p \le 2n}} \log p > \frac{\log 4}{3}\, n - \frac{\sqrt{2n}+3}{2}\, \log(2n).$$

This implies that

$$\sum_{\substack{p \text{ prime} \\ n < p \le 2n}} \log p \ge \frac{1}{3}\, n \tag{5.9.1}$$

for all $n \ge 2349$, which implies Bertrand's postulate in this range. (This lower bound should be compared to the upper bound (5.5.4).)

If $1 \le n \le 5000$, then the interval $(n, 2n]$ contains at least one of the primes 2, 3, 5, 7, 13, 23, 43, 83, 163, 317, 631, 1259, 2503, and 5003. □

Exercise 5.9.2. Use Bertrand's postulate to prove that there are infinitely many primes with first digit "1".

Exercise 5.9.3. Use Bertrand's postulate to show, by induction, that every integer $n > 6$ can be written as the sum of distinct primes.

Exercise 5.9.4. Goldbach conjectured that every even integer ≥ 6 can be written as the sum of two primes. Deduce Bertrand's postulate from Goldbach's conjecture.

Exercise 5.9.5. Use Bertrand's postulate to prove that $\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$ is never an integer.

Exercise 5.9.6. Prove that for every $n \ge 1$ one can partition the set of integers $\{1, 2, \ldots, 2n\}$ into pairs $\{a_1, b_1\}, \ldots, \{a_n, b_n\}$ such that each sum $a_j + b_j$ is a prime.

Exercise 5.9.7.† (a) Prove that prime p divides $\binom{2n}{n}$ when $n/2 < p \le 2n/3$.
(b) Prove that the product of the primes in $(3m, 12m]$ divides $\binom{12m}{6m}\binom{6m}{4m}$.
(c)† Deduce that we can take any constant $c_2 > \frac{2}{9}\log(432)$ in (5.5.1).
(Note that $\frac{2}{9}\log(432) = 1.3485\ldots < \log 4 = 1.3862\ldots$.)
(d) Now deduce Bertrand's postulate for all sufficiently large x from (5.5.1).

5.10. The theorem of Sylvester and Schur

Bertrand's postulate can be rephrased to state that at least one of the integers $k+1, k+2, \ldots, 2k$ has a prime factor $> k$. This can be generalized as follows:

Theorem 5.7 (Sylvester-Schur Theorem)**.** *For any integers $n \ge k \ge 1$, at least one of the integers $n+1, n+2, \ldots, n+k$ is divisible by a prime $p > k$.*

Proposition 5.10.1. *If, for given integers $n \ge k \ge 1$, we have*

$$\binom{n+k}{k} > (n+k)^{\pi(k)}, \tag{5.10.1}$$

then at least one of the integers $n+1, n+2, \ldots, n+k$ is divisible by a prime $p > k$. If (5.10.1) holds for $n_1(k)$, then it holds for all $n \ge n_1(k)$.

Proof. If the prime factors of $n+1, n+2, \ldots, n+k$ are all $\leq k$, then all of the prime factors p of $\binom{n+k}{k}$ are $\leq k$. If $p^e \| \binom{n+k}{k}$, then $p^e \leq n+k$ by Corollary 3.10.1. Therefore

$$\binom{n+k}{k} \leq \prod_{p \leq k} (n+k) = (n+k)^{\pi(k)}, \tag{5.10.2}$$

contradicting (5.10.1). This proves the first part of the result.

We prove the second part by induction on $n \geq n_1(k)$ using the following result.

Exercise 5.10.1. Prove that $\left(1 + \frac{1}{x+k}\right)^k \leq \left(1 + \frac{k}{x+1}\right)$ for all $x \geq k \geq 1$.

The result holds for $n = n_1(k)$, so now suppose that (5.10.1) holds for some given n. Then

$$\binom{n+1+k}{k} = \left(1 + \frac{k}{n+1}\right)\binom{n+k}{k} > \left(1 + \frac{1}{n+k}\right)^k (n+k)^{\pi(k)} > (n+1+k)^{\pi(k)},$$

by exercise 5.10.1 and the induction hypothesis, and so (5.10.1) holds for $n+1$. The result follows. □

Proof of the Sylvester-Schur Theorem for all $k \leq 1500$. Calculations give some value for $n_1(k)$ in Proposition 5.10.1 for all $k \leq 1500$, and so the Sylvester-Schur Theorem follows for these k and all $n \geq n_1(k)$ by Proposition 5.10.1. Now $n_1(k) = k$ for $202 \leq k \leq 1500$, and $k \leq n_1(k) \leq k + 17$ for all $k \leq 201$. We verify the theorem for $k \leq n \leq k + 16$ with $k \leq 201$, case by case. □

A just failed proof of the Sylvester-Schur Theorem. Calculations suggest that $\binom{2k}{k} > (2k)^{\pi(k)}$ for all $k \geq 202$. If so, the Sylvester-Schur Theorem follows for all $k \geq 202$ by Proposition 5.10.1. However we just failed to prove this inequality as a consequence of the upper bound in Theorem 5.3. If one combines the upper bound on $\pi(k/4)$ from Theorem 5.3, together with exercise 5.9.7(b), then we can prove that $\binom{2k}{k} > (2k)^{\pi(k)}$ for all sufficiently large k. However "sufficiently large" here is likely to be extremely large. □

Exercise 5.10.2. Prove that if $\pi(k) < \frac{k \log 4}{\log(2k)} - 1$ for all integers $k \geq 1$, then Theorem 5.7 holds for all $n \geq k \geq 1$.

Proof of the Sylvester-Schur Theorem for all $k > 1500$. If (5.10.1) holds, then the result follows from Proposition 5.10.1. Hence we may assume that (5.10.2) holds. Now, $\pi(k) < k/3$ (which can be proved by accounting for divisibility by 2 and 3), and $\frac{n+k-j}{k-j} > \frac{n+k}{k}$ for $j = 0, \ldots, k-1$ so that $\binom{n+k}{k} \geq (\frac{n+k}{k})^k$. Therefore (5.10.2) implies that

$$\left(\frac{n+k}{k}\right)^k \leq \binom{n+k}{k} \leq (n+k)^{\pi(k)} \leq (n+k)^{k/3},$$

which in turn implies that

$$n + k \leq k^{3/2}; \text{ that is, } n \leq k^{3/2} - k.$$

Next we note that if $p > (n+k)^{1/2}$ and $p^e \| \binom{n+k}{k}$ so that $p^e \leq n+k$, then $e = 0$ or 1. Therefore we can refine (5.10.2) to

$$\binom{n+k}{k} \leq \prod_{p \leq (n+k)^{1/2}} (n+k) \prod_{p \leq k} p = k^{\frac{1}{3}k^{3/4}} 4^{k-1}, \tag{5.10.3}$$

by (5.5.4), as $\pi((n+k)^{1/2}) \leq \frac{1}{3}(n+k)^{1/2} \leq \frac{1}{3}k^{3/4}$.

Now if $n \geq 3k$, then, by exercise 4.14.2 of appendix 4D,

$$\frac{(4^4/3^3)^k}{ek} \leq \binom{4k}{k} \leq \binom{n+k}{k} \leq k^{\frac{1}{2}k^{3/4}} 4^{k-1}$$

which is false for all $k \geq 1$. Therefore $n + k \leq 4k$, and so if $n + k > \frac{5}{2}k$, then our inequality becomes

$$\frac{(5^5/3^3 2^2)^{k/2}}{ek} \leq \binom{5k/2}{k} \leq \binom{n+k}{k} \leq (4k)^{k^{1/2}} 4^{k-1}.$$

This is false for all $k \geq 780$.

Finally for the range $k \leq n \leq 3k/2$ if prime p is in the range $(n+k)/3 < p \leq k$, then $2p$ is the only multiple of p that appears in $(n+1)\cdots(n+k)$ and so p does not divide $\binom{n+k}{k}$. Therefore

$$\binom{2k}{k} \leq \binom{n+k}{k} \leq \prod_{p \leq (n+k)^{1/2}} (n+k) \prod_{p \leq (n+k)/3} p \leq \prod_{p \leq (n+k)^{1/2}} (3k)^{\pi(2k^{1/2})} \prod_{p \leq 5k/6} p,$$

which implies that

$$\frac{4^k}{ek} \leq (4k)^{k^{1/2}} 4^{5k/6-1}$$

which is false for all $k \geq 1471$. □

Exercise 5.10.3. (a) Use Bertrand's postulate and the Sylvester-Schur Theorem to show that if $1 \leq r < s$, then there is a prime p that divides exactly one of the integers $r+1, \ldots, s$.

(b) Deduce that if $1 \leq r < s$, then $\frac{1}{r+1} + \cdots + \frac{1}{s}$ is never an integer.