Appendix 11A. Uniform distribution

11.7. $n\alpha$ mod 1

Dirichlet's Theorem, in section 11.1, implies that $n\alpha$ mod 1 gets arbitrarily close to 0 as n runs through a sequence of integers n. One might also ask whether $n\alpha$ mod 1 gets arbitrarily close to any given $\theta \in (0, 1)$.

Theorem 11.5 (Kronecker's Theorem)**.** *If α is a real irrational number, then the numbers $\{n\alpha\}$ are dense on $[0, 1)$.*

Proof. Fix $\epsilon > 0$. By Dirichlet's Theorem there exists an integer n with $\|n\alpha\| < \epsilon$, where $\|t\|$ is the distance from t to the nearest integer. As α is irrational we also have that $\|n\alpha\| \neq 0$, and so $\{n\alpha\} \in (0, \epsilon)$ *or* $\{n\alpha\} \in (1 - \epsilon, 1)$. We will assume that $\{n\alpha\} \in (0, \epsilon)$ (the case with $\{n\alpha\} \in (1 - \epsilon, 1)$ being proved analogously).

Let $\delta = \{n\alpha\} \in (0, \epsilon)$. Select D to be the largest integer $< 1/\delta$ and so

$$\{n\alpha\},\ \{2n\alpha\}, \ldots, \{Dn\alpha\} = \delta, 2\delta, \ldots, D\delta$$

is a set of points in $[0, 1)$, consecutive points being spaced $\delta < \epsilon$ apart. Therefore if $\theta \in [0, 1)$, then we let $k = [\theta/\delta]$ and so $\theta - k\delta \in [0, \delta)$, which implies that

$$\theta - \{kn\alpha\} = \theta - k\{n\alpha\} = \theta - k\delta \in [0, \delta) \subset [0, \epsilon).$$

That is, there are integer multiples of α in $\mathbb{R}/\mathbb{Z}$ that are arbitrarily close to θ. □

Exercise 11.7.1. Show that the conclusion of the theorem is not true if α is rational.

Exercise 11.7.2. Prove Kronecker's Theorem when $n\alpha \pmod 1 \in (1 - \epsilon, 1)$.

Now we know that if α is irrational, then $n\alpha$ mod 1 gets arbitrarily close to any given $\theta \in [0, 1)$, we might ask how often $n\alpha$ mod 1 gets close to each $\theta \in [0, 1)$. Are the values of $n\alpha$ mod 1 roughly equidistributed? To answer this question we must determine how often $\{n\alpha\} \in [\theta - \epsilon, \theta + \epsilon]$ for $\theta \in (0,1)$ and sufficiently small $\epsilon > 0$. If the numbers $\{n\alpha\}$ are equidistributed, then we might expect the frequency to be roughly proportional to the length of the interval. The analogous question can be asked for any sequence of numbers $x_1, x_2, \ldots \in [0,1)$. We say that $\{x_n\}_{n\geq 1}$ is *uniformly distributed* mod 1 (or *equidistributed* mod 1) if for any $a < b \in [0,1)$,

$$\lim_{N\to\infty} \frac{1}{N} \#\{n \leq N : a \leq x_n \leq b\} \text{ exists and equals } b - a.$$

The values of x (mod 1) are in 1-to-1 correspondence with the values of $e(x)$ (where $e(t) := e^{2i\pi t}$) as its value depends on x (mod 1) and not on x. Moreover the values $e(kx)$ for any given integer $k \neq 0$ remain consistent for x with any given value mod 1. That is, if $x = m + \delta$ with $0 \leq \delta < 1$, then $kx = km + k\delta$ so that $\{kx\} = \{k\delta\}$. This suggests that to study a sequence of values x_n mod 1, we might use Fourier analysis. This thinking leads to the famous theorem of Hermann Weyl (for more on this, including the proof, see [**GG**]):

Theorem 11.6 (Weyl's uniform distribution theorem). *The sequence $\{x_n\}_{n\geq 1}$ is uniformly distributed* mod 1 *if and only if for all non-zero integers k we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e(kx_n) \text{ exists and equals } 0.$$

Exercise 11.7.3. (a) Show that $\sum_{n=1}^{N} e(\{n\alpha\}) = \frac{e(N\alpha)-1}{1-e(-\alpha)}$ if $\alpha \notin \mathbb{Z}$, and then deduce that $|\sum_{n=1}^{N} e(\{n\alpha\})| \leq \frac{1}{|\sin \pi\alpha|}$.

(b) Use Weyl's uniform distribution theorem to deduce that if α is a real, irrational number, then $\{n\alpha\}_{n\geq 1}$ is uniformly distributed mod 1.

One can prove that $\{n\alpha\}$ is uniformly distributed mod 1 using fairly elementary ideas though it is not easy:

Exercise 11.7.4. Let $x_1, x_2, \ldots \in [0,1)$ be a sequence of numbers. Suppose that there are arbitrarily large integers M for which

$$\lim_{N\to\infty} \frac{1}{N} \# \left\{ n \leq N : \frac{m}{M} \leq x_n \leq \frac{m+1}{M} \right\} \text{ exists and equals } \frac{1}{M},$$

for $0 \leq m \leq M - 1$. Deduce that $\{x_n\}_{n\geq 1}$ is uniformly distributed mod 1.

Exercise 11.7.5.‡ Let α be a real, irrational number. In this exercise we sketch a proof that $\{n\alpha\}_{n\geq 1}$ is uniformly distributed mod 1. Fix $\epsilon > 0$ arbitrarily small.

(a) Use Kronecker's Theorem to show that there exists an integer $N \geq 1$ such that $\{N\alpha\} = \delta \in (0, \epsilon)$.

(b) Prove that if $\{n\alpha\} < 1 - \delta$, then $\{(n+N)\alpha\} = \{n\alpha\} + \delta$. What if $\{n\alpha\} \geq 1 - \delta$?

(c) Suppose that $0 < t < 1 - 2\delta$. Show that $\{n\alpha\} \in [t, t+\delta]$ if and only if $\{(n+N)\alpha\} \in [t+\delta, t+2\delta]$, and so deduce that

$$|\#\{1 \leq n \leq x : t \leq \{n\alpha\} < t + \delta\} - \#\{1 \leq n \leq x : t + \delta \leq \{n\alpha\} < t + 2\delta\}| \leq N.$$

Now let $\delta = 1/M$ for some large integer M.

(d)† Use (c) to show that if $0 \leq m \leq M - 1$, then

$$\left| \# \left\{ 1 \leq n \leq x : \frac{m}{M} \leq \{n\alpha\} < \frac{m+1}{M} \right\} - \frac{x}{M} \right| \leq MN.$$

(e) Deduce that $\{n\alpha\}_{n\geq 1}$ is uniformly distributed mod 1 using exercise 11.7.4.

Kronecker's Theorem in n dimensions. In exercise 11.1.2 we saw that Dirichlet's Theorem may be generalized to k dimensions; that is, given $\alpha_1, \dots, \alpha_k \in \mathbb{R}$, for any $\epsilon > 0$ there exist infinitely many integers n such that each $\|n\alpha_j\| < \epsilon$. To generalize Kronecker's Theorem we would like that for $\theta_1, \dots, \theta_k \in \mathbb{R}$ there are infinitely many n for which each $\|n\alpha_j - \theta_j\| < \epsilon$. However this is not true in all cases, even when $k = 1$: In the hypothesis of Theorem 11.5 we needed that α is irrational, and we showed that this is necessary in exercise 11.7.1. Another way to state that α is irrational is to insist that 1 and α are linearly independent over $\mathbb{Z}$.

In two dimensions we find another obstruction: Suppose that $\alpha_1 = \alpha$ and $\alpha_2 = 1 - \alpha$. If $\|n\alpha_j - \theta_j\| < \epsilon$ for each j, then

$$\|\theta_1 + \theta_2\| = \|n - \theta_1 - \theta_2\| \le \|n\alpha_1 - \theta_1\| + \|n\alpha_2 - \theta_2\| < 2\epsilon.$$

But this should hold for any $\epsilon > 0$ which implies that $\theta_1 + \theta_2$ is an integer. Notice that in this example $1, \alpha_1, \alpha_2$ are not linearly independent over $\mathbb{Z}$.

Exercise 11.7.6. Let $\alpha_1, \dots, \alpha_k, \theta_1, \dots, \theta_k \in \mathbb{R}$ be given, and assume that there are integers $c_0, \dots, c_k$ for which $c_0 + c_1\alpha_1 + \cdots + c_k\alpha_k = 0$. Suppose that for all $\epsilon > 0$ there are infinitely many n for which $\|n\alpha_j - \theta_j\| < \epsilon$ for $j = 1, 2, \dots, k$. Prove that $c_1\theta_1 + \cdots + c_k\theta_k \in \mathbb{Z}$.

These are the only obstructions to the generalization:

Theorem 11.7 (Kronecker's Theorem in n dimensions)**.** *Assume that the real numbers $1, \alpha_1, \dots, \alpha_k$ are linearly independent over $\mathbb{Z}$. Then the points*

$$(n\alpha_1, \dots, n\alpha_k)_{n \ge 1} \text{ are dense in } (\mathbb{R}/\mathbb{Z})^k.$$

In other words, for any given $\theta_1, \dots, \theta_k \in \mathbb{R}$ and any $\epsilon > 0$ there are infinitely many integers n for which $\|n\alpha_j - \theta_j\| < \epsilon$ for all $j = 1, \dots, k$.

This can be proved in several different ways that are accessible though tough. We refer the reader to sections 23.5–23.8 of [**HW08**].

11.8. Bouncing billiard balls

Billiards, snooker, and pool are all played on a rectangular table, hitting the ball along the surface. The sides of the table are cushioned so that the ball bounces off the side at the opposite angle to which it hits. That is, if it hits at angle $\alpha°$, then it bounces off at angle $(180 - \alpha)°$. Sometimes one miscues and the ball carries on around the table, coming to a stop without hitting another ball. Have you ever wondered what would happen if there were no friction, so that the ball never stops? Would your ball eventually hit the ball it is supposed to hit, no matter where that other ball is placed? Or could it go on bouncing forever without ever getting to the other ball? We could rephrase this question more mathematically by supposing that we play on a table in the complex plane, with two sides along the x- and y-axes. Say the table length is ℓ and width is w so that it is the rectangle with corners at $(0,0), (0,\ell), (w,0), (w,\ell)$. Let us suppose that the ball is hit from the point (u,v) along a line with slope α (that is, at an angle α from the horizontal).

As the line continues on indefinitely inside the box, does it get arbitrarily close to every point inside the box?

Exercise 11.8.1. Show that by rescaling with the map $x \to x/\ell$, $y \to y/w$ we can assume, without any loss of generality, that the billiards table is the unit square.

As a consequence of exercise 11.8.1, we may henceforth assume that $w = \ell = 1$.

The ball would run along the line $\mathcal{L} := \{(u+t, v+\alpha t),\ t \geq 0\}$ if it did not hit the sides of the table. Notice though that if after each time it hit a side, we reflected the true trajectory through the line that represents that side, then indeed the ball's trajectory would be $\mathcal{L}$.

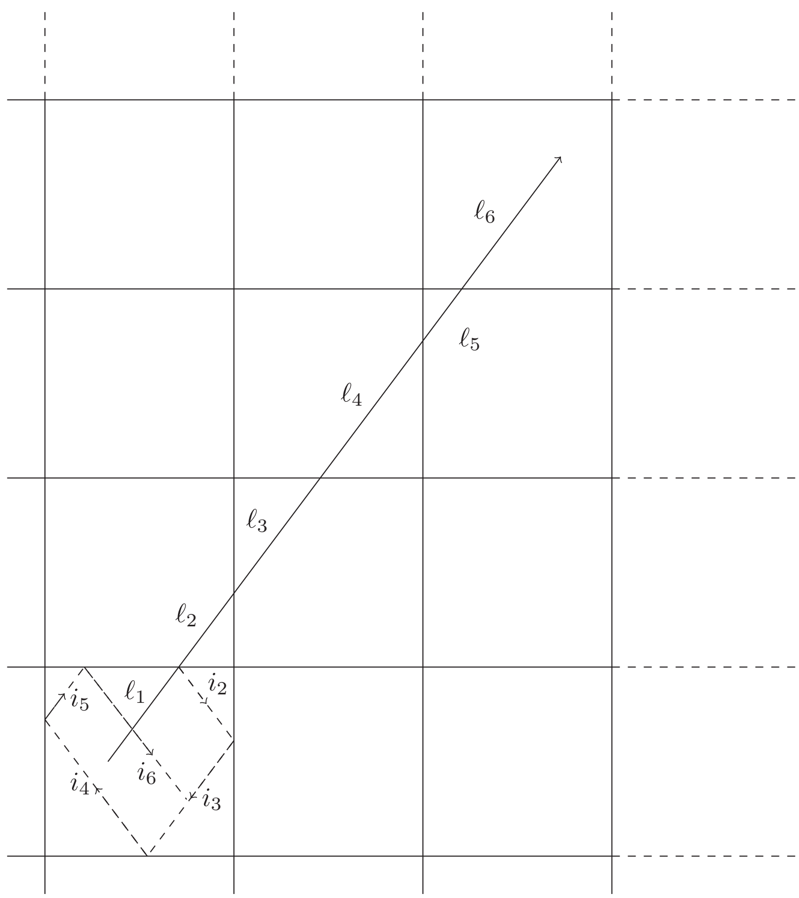

Figure 11.1. Billiards on the complex plane and on the unit square. Following a path inside the fundamental domain of a lattice: The path segment ℓ_j gets mapped to i_j for $j = 2, \ldots, 6$.

Develop this to prove:

Exercise 11.8.2. Show that the billiard ball is at (x, y) after time t, where x and y are given as follows:
Let $m = [u + t]$. If m is even, let $x = \{u + t\}$; if m is odd, let $x = 1 - \{u + t\}$.
Let $n = [v + \alpha t]$. If n is even, let $y = \{v + \alpha t\}$; if n is odd, let $y = 1 - \{v + \alpha t\}$.

Exercise 11.8.3. Show that if α is rational, then the ball eventually ends up exactly where it started from, and so it does not get arbitrarily close to every point on the table.

So how close does the trajectory get to the point (r, s), where $r, s \in [0, 1)$? Let us consider all of those values of t for which $x = r$, with m and n even to simplify matters (with m and n as in exercise 11.8.2), and see if we can determine whether y is ever close to s.

Exercise 11.8.4. Show that $[z]$ is even if and only if $\{z/2\} \in [0, 1/2)$. Deduce that $[z]$ is even and $\{z\} = r$ if and only if $\{z/2\} = r/2$.

Hence we want $(u+t)/2 = k+r/2$ for some integer k; that is, $t = 2k+(r-u)$, $k \in \mathbb{Z}$. In that case $v + \alpha t = 2\alpha k + \alpha(r - u) + v$ so we want $\{\alpha k + (\alpha(r - u) + v)/2\}$ close to $s/2$. That is, $k\alpha$ mod 1 should be close to $\theta := \{\frac{(s-v)+\alpha(u-r)}{2}\}$. Now, in Kronecker's Theorem (Theorem 11.5) we showed that the values $k\alpha$ mod 1 are dense in $[0, 1)$ when α is irrational, and so in particular there are values of k that allow $k\alpha$ mod 1 to be arbitrarily close to θ. Hence we have proved the difficult part of the following corollary:

Corollary 11.8.1. *If α is a real irrational number, then any ball moving at angle α (to the coordinate axes) will eventually get arbitrarily close to any point on a* 1*-by-*1 *billiards table.*

We finish with a challenge question to develop a similar theory of billiards played on a circular table!

Exercise 11.8.5. Imagine a trajectory inside the unit circle. A ball is hit and continues indefinitely. When it hits a side at angle θ (compared to the normal line at that point), it bounces off at angle $-\theta$.

(a) Suppose that the first two points at which the ball hits the edge are at $e(\beta)$ and then at $e(\beta + \alpha)$. Show that the ball hits the edge at $e(\beta + n\alpha)$ for $n = 0, 1, 2, \ldots$.
(b) Prove that the ball falls into a repeated trajectory if and only if α is rational.
(c) Show that if α is irrational, then the points at which the ball hits the circle edge are dense (i.e., eventually the ball comes arbitrarily close to any point on the edge) but that it never hits the same edge point twice.
(d) Prove that the ball's trajectory never comes inside the circle of radius $|\cos(\alpha/2)|$. Deduce that the trajectory of the ball is *never* dense inside the unit circle.
(e) Prove that if α is irrational, then the trajectory of the ball is dense inside the ring between the circle of radius $|\cos(\alpha/2)|$ and the circle of radius 1. (The technical word for a ring is an *annulus*.)

Appendices. The extended version of chapter 11 has the following additional appendices:

Appendix 11B. *Continued fractions* introduces and analyzes continued fractions for all real numbers, focusing on continued fractions for quadratic irrationals. We find and justify a particularly efficient algorithm for finding all the solutions to Pell's equation using continued fractions.

Appendix 11C. *Two-variable quadratic equations* establishes that, other than in certain special cases, if there is one solution to a given two-variable quadratic equation, then there are infinitely many.

Appendix 11D. *Transcendental numbers* discusses how many transcendental numbers there are, via Cantor's diagonalization argument. We show that e and π are irrational and then discuss "normal numbers".