Appendix 0A. A closed formula for sums of powers

In chapter 0, we discussed closed form expressions for sums of powers. We will prove here that there is such a formula for the sum of the kth power of the integers up to a given point, developing themes from earlier in this chapter.

0.5. Formulas for sums of powers of integers, II

Our goal in this section is to use our formula (0.3.2) for summing binomial coefficients, to find a formula for summing powers of integers. For example, since

$$n^3 = 6\binom{n}{3} + 6\binom{n}{2} + \binom{n}{1},$$

we can use (0.3.2) with $k = 3$, 2, and 1, respectively, to obtain

$$\begin{aligned} \sum_{n=0}^{N-1} n^3 &= 6\sum_{n=0}^{N-1}\binom{n}{3} + 6\sum_{n=0}^{N-1}\binom{n}{2} + \sum_{n=0}^{N-1}\binom{n}{1} \\ &= 6\binom{N}{4} + 6\binom{N}{3} + \binom{N}{2}. \end{aligned}$$

Summing these three multiples of binomial coefficients gives the formula for the sum of the cubes of the integers up to $N-1$, which we encountered in section 0.2. To make this same technique work to sum n^k, for arbitrary integer $k \geq 1$, we need to show that x^k can be expressed as a sum of fixed multiples of the binomial coefficients $\binom{x}{k}, \ldots, \binom{x}{1}$, where by $\binom{x}{k}$ we mean the polynomial

$$\binom{x}{k} = \frac{x(x-1)\cdots(x-(k-1))}{k!}.$$

Notice that if we substitute $x = n$ into this expression, we obtain the binomial coefficient $\binom{n}{k}$.

Proposition 0.5.1. *Any polynomial $f(x) \in \mathbb{Z}[x]$ of degree $k \geq 0$ can be written as a sum of integer multiples of the binomial coefficients $\binom{x}{k}, \ldots, \binom{x}{1}, \binom{x}{0}$.*

Proof. By induction on k. The result is immediate for $k = 0$. Otherwise, suppose that $f(x)$ has leading coefficient ax^k; then subtract $a \cdot k! \cdot \binom{x}{k}$, which also has leading coefficient ax^k. The resulting polynomial, $g(x) = f(x) - a \cdot k! \cdot \binom{x}{k}$, has degree $k-1$ so can be written as $c_0\binom{x}{0} + \cdots + c_{k-1}\binom{x}{k-1}$ by the induction hypothesis. But then $f(x) = c_0\binom{x}{0} + \cdots + c_k\binom{x}{k}$, with $c_k = a \cdot k!$, as desired. □

In particular, there are integers $c_0,\ c_1, \ldots, c_k$ for which

$$x^k \;=\; c_k\binom{x}{k} + \cdots + c_1\binom{x}{1} + c_0\binom{x}{0}. \tag{0.5.1}$$

One can then immediately deduce, from (0.3.2), that

$$\begin{aligned} \sum_{n=0}^{N-1} n^k = c_k \sum_{n=0}^{N-1}\binom{n}{k} + \cdots + c_1\sum_{n=0}^{N-1}\binom{n}{1} + c_0\sum_{n=0}^{N-1}\binom{n}{0} \\ = c_k\binom{N}{k+1} + \cdots + c_1\binom{N}{2} + c_0\binom{N}{1}. \end{aligned}$$

Expanding out the binomial coefficients, this gives the desired closed form expression for $\sum_{n=0}^{N-1} n^k$, a polynomial in N of degree $k+1$.

There is a difficulty. We proved that the c_j exist but did not show how to determine them. We can do this by successively substituting in $x = 0$, then $x = 1$, then $\ldots, x = k-1$ into (0.5.1), since one obtains

$$0^k = c_k \cdot 0 + \cdots + c_1 \cdot 0 + c_0,$$

and so $c_0 = 0$; then

$$1^k \;=\; c_k \cdot 0 + \cdots + c_2 \cdot 0 + c_1 + c_0,$$

and so $c_1 = 1$; and then $c_2 = 2^k - 2$, $c_3 = 3^k - 3 \cdot 2^k + 3$, etc. We end this appendix with a particularly challenging exercise.

Exercise 0.5.1.‡ (a) Establish that (0.5.1) holds with

$$c_m \;=\; m^k - \binom{m}{1}(m-1)^k + \binom{m}{2}(m-2)^k - \cdots + (-1)^{m-2}\binom{m}{m-2}2^k + (-1)^{m-1}m,$$

for all $m \geq 1$ and for all $k \geq 1$. The integers $c_m/m!$ are the *Stirling numbers of the second kind*, usually denoted by $S_2(k,m)$. They arise in several interesting combinatorial settings; for example, $S_2(k,m)$ is the number of ways to partition a set of k objects into m non-empty subsets.

(b) Deduce that, for any given integer $k \geq 0$, there exist rational numbers $a_0, a_1, \ldots, a_{k+1}$ for which $\sum_{n=0}^{N-1} n^k = a_0 + a_1 N + \cdots + a_{k+1}N^{k+1}$ for all integers $N \geq 1$.

Exercise 0.5.2. Prove that $c_j/j!$ is an integer for all $j \geq 0$ in (0.5.1).

Exercise 0.5.3.† Let $f(x) \in \mathbb{C}[x]$. Prove that $f(n)$ is an integer for all integers n if and only if $f(x) = \sum_m a_m\binom{x}{m}$ where the a_m are all integers.

We will return to this topic, finding an elegant description of the rational numbers a_j by introducing the Bernoulli numbers in the next appendix, appendix 0B.