At the moment the function $\log_p(z)$ is defined only when $|z-1|_p < 1$. For any $\beta \in \mathbb{Z}_p$ with $|\beta|_p = 1$, there exists an integer b with $b \equiv \beta \pmod p$ and $b \not\equiv 0 \pmod p$. Therefore, by Fermat's little theorem, $\beta^{p-1} \equiv b^{p-1} \equiv 1 \pmod p$, and so $\log_p(\beta^{p-1})$ is well-defined. Taking our lead from exercise 16.6.5(c) we therefore define

$$\log_p(\beta) \ := \ \frac{\log_p(\beta^{p-1})}{p-1} \ = \ \lim_{k\to\infty} \frac{\beta^{p^k(p-1)}-1}{p^k(p-1)}.$$

Exercise 16.6.6. Assume that $\alpha, \beta \in \mathbb{Z}_p$.
(a) Prove that $\log_p(-\alpha) = \log_p(\alpha)$.
(b) Prove that $\log_p(\alpha\beta) = \log_p(\alpha) + \log_p(\beta)$.

Any $\gamma \in \mathbb{Z}_p$ can be written in the form $\gamma = p^e\beta$ where $|\beta|_p = 1$, so we define[3]

$$\log_p(\gamma) \ = \ e\log_p(p) + \log_p(\beta).$$

16.7. The p-adic dilogarithm

For each $k \geq 1$, define

$$\mathcal{L}_k(x) := \sum_{m\geq 1} \frac{x^m}{m^k}.$$

The case $k=2$ is the *dilogarithm function.*

Exercise 16.7.1. (a) Prove that the sum defining $\mathcal{L}_k(x)$ converges for all $x \in \mathbb{C}$ with $|x|_\infty \leq 1$ for all $k \geq 2$, and for $|x|_p < 1$ in the p-adics.
(b) Establish that $\mathcal{L}_k(x) + \mathcal{L}_k(-x) = 2^{1-k}\mathcal{L}_k(x^2)$ when $|x|_p < 1$.

Theorem 16.5. *If* $|1-z|_p < 1$ *then*

$$\mathcal{L}_2(1-z) + \mathcal{L}_2(1-z^{-1}) \ = \ -\frac{1}{2}(\log_p z)^2. \tag{16.7.1}$$

In particular $\mathcal{L}_2(2) = 0$ *in the* 2*-adics.*

Proof. For $|x|_p < 1$, we have

$$\frac{\mathrm{d}\mathcal{L}_2(x)}{\mathrm{d}x} \ = \ \frac{1}{x}\sum_{m\geq 1}\frac{x^m}{m} = -\frac{\log_p(1-x)}{x} \ ,$$

and so, by the chain rule, we have

$$\begin{aligned}\frac{d}{dz}(\mathcal{L}_2(1-z) + \mathcal{L}_2(1-z^{-1})) \ &= \ -\mathcal{L}_2'(1-z) + z^{-2}\mathcal{L}_2'(1-z^{-1}) \\ &= \ \frac{\log_p(z)}{1-z} - z^{-2}\frac{\log_p(z^{-1})}{1-z^{-1}} \ = \ -\frac{\log_p z}{z}.\end{aligned}$$

Integrating yields $\mathcal{L}_2(1-z) + \mathcal{L}_2(1-z^{-1}) = -\frac{1}{2}(\log_p z)^2 + C$ for some constant C. Taking $z=1$ we see that $C=0$, yielding (16.7.1).

Replacing z by z^2, we obtain

$$\mathcal{L}_2(1-z^2) + \mathcal{L}_2(1-z^{-2}) \ = \ -2(\log_p z)^2 \ = \ 4(\mathcal{L}_2(1-z) + \mathcal{L}_2(1-z^{-1})).$$

When $p=2$ we may take $z=-1$ in this equation and so $8\mathcal{L}_2(2) = 2\mathcal{L}_2(0) = 0$. □

Exercise 16.7.2. Let $p=2$ and $|z-1|_2 < 1$.

[3]We can select any value for $\log_p(p)$; we do not have to let it be 1.

(a) Prove that $\mathcal{L}_2(1-z)+\mathcal{L}_2(1+z)=\frac{1}{2}\mathcal{L}_2(1-z^2)+C$ for some constant C.
(b) Prove that $C=0$ using (16.7.1).
(c) Deduce (again) that $\mathcal{L}_2(2)=0$.

We have now seen that
$$\sum_{n\geq 1}\frac{2^n}{n}=\sum_{n\geq 1}\frac{2^n}{n^2} \;=\; 0$$
in the 2-adics. It is interesting to see how rapidly this convergence happens. If $n\geq N\geq 2^k$ then $v_2(2^n/n)\geq 2^k-k$ so that
$$\sum_{n<N}\frac{2^n}{n}=-\sum_{n\geq N}\frac{2^n}{n}\equiv 0 \pmod{2^{2^k-k}}$$
and similarly
$$\sum_{n<N}\frac{2^n}{n^2}\equiv 0 \pmod{2^{2^k-2k}}.$$
It looks like there might be a pattern here. How about $\sum_{n\geq 1}2^n/n^3$? Unfortunately the $n=4$ term gives the unique maximum, 2^2, of $|2^n/n^3|_2$, and so $|\sum_{n\geq 1}2^n/n^3|_2=4$, not 0.

Exercise 16.7.3. Prove that if $|x|_p,|y|_p<1$ then
$$\mathcal{L}_2(x)+\mathcal{L}_2(y)-\mathcal{L}_2(xy)-\mathcal{L}_2\left(\frac{x(1-y)}{1-xy}\right)-\mathcal{L}_2\left(\frac{y(1-x)}{1-xy}\right)=\log_p\left(\frac{1-x}{1-xy}\right)\log_p\left(\frac{1-y}{1-xy}\right).$$

Further reading on p-adics

[1] Richard M. Hill, *Introduction to number theory*, World Scientific, Singapore (2018), chapter 4.