Concours Putnam

Atelier de Pratique

Le jeudi, 13 novembre 13h30-14h30 (Salle: Pavillon André-Aisenstadt 5448)

Induction mathématique

1. Prove Bernoulli's inequality, which states that if $x > -1$, $x \neq 0$ and n is a positive integer greater than 1, then $(1+x)^n > 1 + nx$.

Solution: By induction. For the base case $n = 2$ the inequality is $(1+x)^2 > 1+2x$, obviously true because $(1+x)^2 - (1+2x) = x^2 > 0$. For the induction step, assume that the inequality is true for n, i.e., $(1+x)^n > 1 + nx$. Then, for $n+1$ we have

$$(1+x)^{n+1} = (1+x)^n(1+x) > (1+nx)(1+x) = 1 + (n+1)x + x^2 > 1 + (n+1)x,$$

and the inequality is also true for $n+1$.

2. Let r be a number such that $r + 1/r$ is an integer. Prove that for every positive integer n, $r^n + 1/r^n$ is an integer.

Solution: We prove it by induction. For $n = 1$ the expression is indeed an integer. For $n = 2$ we have that $r^2 + 1/r^2 = (r + 1/r)^2 - 2$ is also an integer. Next assume that $n > 2$ and that the expression is an integer for $n-1$ and n. Then we have

$$\left(r^{n+1} + \frac{1}{r^{n+1}}\right) = \left(r^n + \frac{1}{r^n}\right)\left(r + \frac{1}{r}\right) - \left(r^{n-1} + \frac{1}{r^{n-1}}\right)$$

hence the expression is also an integer for $n+1$.

3. Find the maximum number $R(n)$ of regions in which the plane can be divided by n straight lines.

Solution: After some experimentation, one can conjecture the formula $R(n) = (n^2 + n + 2)/2$. We will prove by induction that it works for all $n \geq 1$. For $n = 1$ we have $R(1) = 2 = (1^2 + 1 + 2)/2$, which is correct. Next assume that the property is true for some positive integer n. Let's look at what happens when we introduce the $(n+1)$th straight line. In general this line will intersect the other n lines in n different intersection points, and it will be divided into $n+1$ segments by those intersection points. Each of those $n+1$ segments divides a previous region into two regions, so the number of regions increases by $n+1$. Hence

$$R(n+1) = R(n) + n + 1.$$

But by induction hypothesis, $R(n) = (n^2 + n + 2)/2$, hence:

$$R(n+1) = \frac{n^2 + n + 2}{2} + n + 1 = \frac{n^2 + 3n + 4}{2} = \frac{(n+1)^2 + (n+1) + 4}{2}$$

4. We divide the plane into regions using straight lines. Prove that those regions can be colored with two colors so that no two regions that share a boundary have the same color.

Solution: We prove it by induction in the number n of lines. For $n = 1$ we will have two regions, and we can color them with just two colors, say one in red and the other in blue. Assume that the plane divided by n lines has been colored in the desired way. After we introduce the $(n+1)$th line we need to recolor the plane to make sure that the new coloring still verifies that no two regions that share a boundary have the same color. We do it in the following way. The $(n+1)$th line divides the plane into two half-planes. We keep intact the colors in all the regions that lie in one half-plane, and reverse the colors (change red to blue and blue to red) in all the regions of the other half-plane. So if two regions share a boundary and both lie in the same half-plane, they will still have different colors. Otherwise, if they share a boundary but are in opposite half-planes, then they are separated by the $(n+1)$th line; which means they were part of the same region, so had the same color, and must have acquired different colors after recoloring.

5. A great circle is a circle drawn on a sphere that is an "equator", i.e., its center is also the center of the sphere. There are n great circles on a sphere, no three of which meet at any point. They divide the sphere into how many regions?

Solution: The answer is $f(n) = n^2 - n + 2$. The proof is by induction. For $n = 1$ we get $f(1) = 2$, which is indeed correct. Then we must prove that if $f(n) = n^2 - n + 2$ then $f(n+1) = (n+1)^2 - (n+1) + 2$. In fact, the $(n+1)$th great circle meets each of the other great circles in two points each, so $2n$ points in total, which divide the circle into $2n$ arcs. Each of these arcs divides a region into two, so the number of regions grow by $2n$ after introducing the $(n+1)$th circle. Consequently $f(n+1) = f(n) + 2n = n^2 - n + 2 + 2n = n^2 + n + 2 = (n+1)^2 - (n+1) + 2$.

6. Prove that every natural number can be represented in the form

$$n = 3^{u_1}2^{v_1} + 3^{u_2}2^{v_2} + \ldots + 3^{u_k}2^{v_k},$$

where $u_1 > u_2 > \ldots > u_k \geq 0$ and $0 \leq v_1 < v_2 < \ldots < v_k$ are integers.

Solution: We proceed by induction.

Base case. For $n = 1$, the statement is obvious.

Inductive step. Suppose $n = 2m$. By the induction hypothesis, the number m can be represented in the required form. By increasing all exponents v_i by 1, we obtain a representation for n.

Now suppose n is odd. Let $v_0 = 0$, and consider the largest integer u_0 such that $3^{u_0} \leq n$. If the inequality is strict, represent the number

$$l = n - 3^{u_0}$$

in the required form. Then $v_1 > 0$, since l is even, and $u_1 < u_0$, because

$$l < 3^{u_0+1} - 3^{u_0} = 2 \cdot 3^{u_0}.$$

Thus, in both cases, the number n can be expressed in the required form.

7. This is a modified version of the game of Nim (in the following we assume that there is an unlimited supply of tokens.) Two players arrange several piles of tokens in a row. By turns each of them takes one token from one of the piles and adds at will as many tokens as he or she wishes to piles placed to the left of the pile from which the token was taken. Assuming that the game ever finishes, the player that takes the last token wins. Prove that, no matter how they play, the game will eventually end after finitely many steps.

Solution: We use induction on the number n of piles. For $n = 1$ we have only one pile, and since each player must take at least one token from that pile, the number of tokes in it will decrease at each move until it is empty. Next, for the induction step, assume that the game with n piles must end eventually. Note that the players cannot keep taking tokens only from the first n piles, since by induction hypothesis the game with n piles eventually ends. So sooner or later one player must take a token from the $(n + 1)$th pile. It does not matter how many tokes he or she adds to the other n piles after that, it is still true that the players cannot keep taking tokens only from the first n piles forever, so eventually someone will take another token from the $(n + 1)$th pile. Consequently, the number of tokens in that pile will continue decreasing until it is empty. After that we will have only n piles left, and by induction hypothesis the game will end in finitely many steps after that.

8. Call an integer square-full if each of its prime factors occurs to a second power (at least). Prove that there are infinitely many pairs of consecutive square-fulls.

Solution: The numbers 8 and 9 are a pair of consecutive square-fulls. Next, if n and $n+1$ are square-full, so are $4n(n+1)$ and $4n(n+1)+1=(2n+1)^2$.

9. The vertices of a convex polygon are colored in three colors so that each color is present and no two adjacent vertices are colored in the same color. Prove that the polygon can be divided by diagonals into triangles such that each triangle has vertices of three different colors.

Solution: Let us denote the three colors by the numbers 1, 2, and 3. We will prove the statement by induction on the number n of vertices of the polygon.

Base case. For $n=3$, the statement is trivial.

Inductive step. Assume $n>3$. Choose two vertices A and B of the same color, say color 1. The points A and B divide the boundary of the polygon into two polygonal chains. On each of these chains, there is at least one vertex of a color different from 1.

We can find two vertices C and D of colors 2 and 3, respectively, such that C and D lie on different chains. Indeed, this is easy if each of the two chains contains vertices of both colors 2 and 3. If, however, one of the chains contains no vertex of, say, color 2, then all its vertices must be of color 3, and therefore on the other chain there is a vertex of color 2.

Draw the diagonal CD and divide the original n-gon into two polygons M_1 and M_2. Each of these polygons satisfies the same conditions as in the problem statement. By the induction hypothesis, each of them can be triangulated so that all triangles have vertices of three different colors. Combining these triangulations gives the desired triangulation of the original polygon.

10. At an Olympiad, 2025 participants arrived, some of whom are acquainted with each other. We say that several pairwise acquainted participants form a *circle* if every other participant of the Olympiad is unacquainted with at least one of them.

Prove that it is possible to seat all participants of the Olympiad into 90 rooms so that in no room will all members of any "circle" be seated together.

Solution: We prove by induction on k a more general statement: $2k$ rooms are sufficient to seat $n \leq k^2$ participants. Then, to obtain the statement of the problem, it will suffice to substitute $k=45$, since $2025=45^2$.

Base case $k=1,2$**.** Since $2k \geq k^2$, we can seat each participant in a separate room.

Inductive step. Assume the statement holds when the number of participants does not exceed $(k-1)^2$. We will prove it when the number of participants does not exceed k^2.

Consider a participant v with the largest number d of acquaintances.

If $d \geq 2k-2$, then seat v in one room, all of their acquaintances in a second room, and the remaining

$$n - 1 - d \leq k^2 - 1 - (2k-2) = (k-1)^2$$

participants can, by the induction hypothesis, be seated in $2(k-1)$ rooms so that there are no "circles" in those rooms. In the first room there is only one person, so no "circle" can appear there. In the second room there are no "circles," since v (who knows everyone in that room) is not there.

If instead $d < 2(k-2)$, note that $d+1 \leq 2k$ rooms will certainly suffice. Take $d+1$ rooms and seat the participants one by one so that no two acquainted participants sit in the same room. Then, within any room, no "circle" can form (since the people sitting together are all unacquainted). Each participant has at most d acquaintances, and since there are $d+1$ rooms, there will always be a room containing none of their acquaintances — we seat them there.

This completes the induction.