

Concours Putnam

Atelier de Pratique

Le jeudi, 4 décembre 13h30-14h30 (Salle: Pavillon André-Aisenstadt 5448)

Linear algebra

1. Let V be the vector space of real polynomials of degree $\leq n$. Define a linear operator $T : V \rightarrow V$ by

$$T(f)(x) = f(x+1) - f(x).$$

Determine the minimal polynomial of T .

2. Let A be an $n \times n$ real matrix such that $A^3 = 0$. Prove that

$$\text{rk}(A) \leq \frac{2n}{3}.$$

Moreover, determine precisely when equality holds.

3. Let A be an $n \times n$ complex (or real) matrix such that

$$\text{tr}(A^k) = 0 \quad \text{for } k = 1, 2, \dots, n.$$

Prove that A is nilpotent.

4. Let P be a $n \times n$ matrix satisfying

$$P^2 = P.$$

(Such matrices are called *idempotent*). Prove that:

- All eigenvalues of P are either 0 or 1.
 - $\text{trace}(P) = \text{rk}(P)$.
5. Twenty-four students solved twenty-five problems. The instructor has a 24×25 table indicating which student solved which problem. It turned out that each problem was solved by at least one student. Prove that it is possible to mark some of the problems with a “checkmark” so that each student solved an even number (possibly zero) of the marked problems.
6. In an $m \times n$ table, numbers are written so that for any two rows and any two columns, the sum of the numbers in one pair of opposite corners of the rectangle they form equals the sum of the numbers in the other pair of opposite corners. Some of the numbers were erased, but the remaining ones allow us to recover all the erased entries. Prove that at least $n + m - 1$ numbers must remain.
7. Several light bulbs on a panel are lit. There are several buttons. Pressing a button changes the state of all bulbs connected to it. It is known that for *any* set of bulbs, there exists a button connected to an odd number of bulbs from that set. Prove that, by pressing the buttons, one can turn off all the bulbs.

8. A herd contains 101 cows. If we remove *any* cow from the herd, the remaining 100 cows can be split into two groups of 50 cows each having the same total weight. Prove that all cows have the same weight.
9. On the segment $[0, 1]$ several distinct points are marked. Each marked point is either exactly in the middle between two other marked points (not necessarily adjacent to it), or exactly in the middle between a marked point and an endpoint of the segment. Prove that all marked points are rational.