Concours Putnam

Atelier de Pratique
Le jeudi, 21 novembre 12h30-13h30
5448 Pav. André Aisenstadt

Fonctions et Integrals

1. Let n be a positive integer, and define

$$f(n) = 1! + 2! + \ldots + n!.$$

   Find polynomials $P(x)$ and $Q(x)$ such that

$$f(n+2) = P(n)f(n+1) + Q(n)f(n)$$

   for all $n \geq 1$.

2. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that $f(nm) = f(n)f(m)$, and such that

$$\lim_{n\to\infty} \frac{\log(f(n))}{\log n} = 1$$

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that for every x,

$$f(x+1) = \frac{1+f(x)}{1-f(x)}$$

   Prove that f is periodic.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous, with $f(x)f(f(x)) = 1$ for all $x \in \mathbb{R}$. If $f(1000) = 999$, find $f(500)$.

5. Compute the following integrals
   1. $\int_0^\infty \frac{xdx}{e^x-1}$
   2. $\int_0^1 \frac{\ln(1+x)}{x}dx$

6. Let $f \in C(0,\infty)$ satisfy $\int_0^1 f^2 < \infty$. Define $\psi(x) = \frac{1}{x}\int_0^x f(t)dt$. Prove that for all $T > 0$,

$$\int_0^T \psi(x)^2 dx \leq 2\int_0^T f(x)\psi(x)dx.$$

7. Prove that $\int_0^1 \frac{dx}{x^x} = 1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \ldots$

8. Prove or disprove the following statement: If F is a finite set with two or more elements, then there exists a binary operation * on F such that for all x, y, z in F,
   1. $x * z = y * z$ implies $x = y$ (right cancellation holds), and
   2. $x * (y * z) \neq (x * y) * z$ (no case of associativity holds).