Concours Putnam

Atelier de Pratique
Le jeudi, 21 novembre 12h30-13h30
5448 Pav. André Aisenstadt

Fonctions et Integrals

1. Let n be a positive integer, and define

$$f(n) = 1! + 2! + \ldots + n!.$$

Find polynomials $P(x)$ and $Q(x)$ such that

$$f(n+2) = P(n)f(n+1) + Q(n)f(n)$$

for all $n \geq 1$.

Solution: This is (Putnam '84 B1) We have

$$f(n+2) - f(n+1) = (n+2)! = (n+2)(n+1)! = (n+2)(f(n+1) - f(n)),$$

hence

$$f(n+2) = (n+2)(f(n+1) - f(n)) + f(n+1) = (n+3)f(n+1) - (n+2)f(n),$$

and we can take $P(x) = x + 3, Q(x) = -x - 2$.

2. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that $f(nm) = f(n)f(m)$, and such that

$$\lim_{n\to\infty} \frac{\log(f(n))}{\log n} = 1$$

Solution: We will see that $f(n) = n$. Suppose the opposite. There is an m such that $\frac{f(m)}{m} = \alpha \neq 1$. Now we take the limit over the subsequence m^k with $k \to \infty$ and m fixed. Thus

$$1 = \lim_{k\to\infty} \frac{\log(f(m^k))}{\log(m^k)} = \frac{\log(\alpha m)}{\log m} = 1 + \frac{\log \alpha}{\log m}$$

and we get a contradiction if α si different from 1.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that for every x,

$$f(x+1) = \frac{1 + f(x)}{1 - f(x)}$$

Prove that f is periodic.

Solution: We have $f(x+2) = -1/f(x)$, hence $f(x+4) = f(x)$, and f is periodic with period 4.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous, with $f(x)f(f(x)) = 1$ for all $x \in \mathbb{R}$. If $f(1000) = 999$, find $f(500)$.

Solution: First notice that $f(1000)f(999) = 1$ implies that $f(999) = \frac{1}{999}$. Since $f(1000) = 999$, and the function is continuous, we conclude that there is a value a for which $f(a) = 500$. Then $f(500)f(a) = 1$ so that $f(500) = \frac{1}{f(a)} = \frac{1}{500}$.

5. Compute the following integrals

   1. $\int_0^\infty \frac{xdx}{e^x - 1}$
   2. $\int_0^1 \frac{\ln(1+x)}{x} dx$

Solution:

1.

$$\int_0^\infty \frac{xdx}{e^x - 1} = \int_0^\infty \frac{xe^{-x}dx}{1 - e^{-x}} = \int_0^\infty xe^{-x} \left(\sum_{k=0}^{\infty} e^{-kx} \right) dx.$$

Now use parts to see that for $\alpha < 0$, $\int_0^\infty xe^{\alpha x} dx = \frac{xe^{\alpha x}}{\alpha}\Big|_0^\infty - \int_0^\infty \frac{e^{\alpha x}}{\alpha} dx = \frac{1}{\alpha^2}$ Thus we obtain

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} = \zeta(2).$$

2.

$$\int_0^1 \frac{\ln(1+x)}{x} dx = \int_0^1 -\sum_{k=1}^{\infty} \frac{(-1)^k x^{k-1}}{k} dx = \sum_{k=1}^{\infty} (-1)^{k-1} \int_0^1 \frac{x^{k-1}}{k} dx$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2}$$

Notice that

$$\zeta(2)+\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}=\sum_{k=1}^{\infty}\frac{1}{k^2}+\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}$$
$$=2\sum_{k=1,k\text{even}}^{\infty}\frac{1}{k^2}=\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{1}{2}\zeta(2)$$

Then

$$\int_0^1\frac{\ln(1+x)}{x}dx=\frac{\zeta(2)}{2}$$

6. Let $f\in C(0,\infty)$ satisfy $\int_0^1 f^2<\infty$. Define $\psi(x)=\frac{1}{x}\int_0^x f(t)dt$. Prove that for all $T>0$,

$$\int_0^T\psi(x)^2dx\leq 2\int_0^T f(x)\psi(x)dx.$$

Solution: Consider

$$I=\int_0^T\psi(x)^2dx=-\int_0^T\left(\frac{1}{x}\right)'\left(\int_0^x f(t)dt\right)^2dx.$$

Integrating by parts we get

$$I=2\int_0^T f(x)\frac{1}{x}\left(\int_0^x f(t)dt\right)dx-\frac{1}{x}\left(\int_0^x f(t)dt\right)^2\Bigg|_0^T.$$

By square integrability of f and Cauchy-Schwartz, we learn that $\left(\int_0^x f(t)dt\right)^2\leq Cx^2$ for $x\leq 1$, and so

$$\frac{1}{x}\left(\int_0^x f(t)dt\right)^2\Bigg|_0^T=-\frac{1}{T}\left(\int_0^T f\right)^2\leq 0.$$

Thus $I\leq 2\int_0^t f(x)\psi(x)dx$.

7. Prove that $\int_0^1\frac{dx}{x^x}=1+\frac{1}{2^2}+\frac{1}{3^3}+\frac{1}{4^4}+\dots$

Solution:
$$\int_0^1 \frac{dx}{x^x} = \int_0^1 e^{-x\ln x}dx = \int_0^1 \sum_{k=0}^{\infty} \frac{(-1)^k x^k \ln^k x}{k!}dx$$

Let $I_k = \int_0^1 x^k \ln^k x dx$. Then

$$I_k = \frac{x^{k+1}}{k+1}\ln^k x\Big|_0^1 - \int_0^1 \frac{kx^k}{k+1}\ln^{k-1} x dx$$
$$= -\frac{kx^{k+1}}{(k+1)^2}\ln^{k-1} x\Big|_0^1 + \int_0^1 \frac{k(k-1)x^k}{(k+1)^2}\ln^{k-2} x dx$$
$$... = (-1)^k \int_0^1 \frac{k!x^k}{(k+1)^k}dx = (-1)^k \frac{k!}{(k+1)^{k+1}}.$$

Thus we get
$$\int_0^1 \frac{dx}{x^x} = \sum_{k=0}^{\infty} \frac{1}{(k+1)^{k+1}}$$

8. Prove or disprove the following statement: If F is a finite set with two or more elements, then there exists a binary operation * on F such that for all x, y, z in F,
   1. $x * z = y * z$ implies $x = y$ (right cancellation holds), and
   2. $x * (y * z) \neq (x * y) * z$ (no case of associativity holds).

Solution: This is (Putnam 84, B3) The statement is true. Let φ any bijection on F with no fixed points ($\varphi(x) \neq x$ for every x), and set $x * y = \varphi(x)$. Then

1. $x * z = y * z$ is equivalent to $\varphi(x) = \varphi(y)$, and this implies $x = y$ because φ is a bijection.
2. We have $x * (y * z) = \varphi(x)$ and $(x * y) * z = \varphi(\varphi(x))$, which cannot be equal because that would imply than $\varphi(x)$ is a fixed point of φ.