Concours Putnam

Atelier de Pratique

Le jeudi, 10 septembre 14h30-15h30

1. Prove that it is impossible to choose three distinct integers a, b and c such that

$$a-b|b-c, \quad b-c|c-a, \quad c-a|a-b.$$

Solution: Let d_1 , d_2 , d_3 be the integers with

$$b-c=d_1(a-b), \quad c-a=d_2(b-c), \quad a-b=d_3(c-a).$$

By multiplying the equations it is easy to show that $d_1d_2d_3 = 1$. Hence, either $d_1 = d_2 = d_3 = 1$ or, say, $d_1 = d_2 = -1$ and $d_3 = 1$.

If $d_1 = d_2 = d_3 = 1$, we easily obtain that $a = b = c$, which contradicts the assumption that a, b, and c are distinct. If $d_1 = -1$ then $a = c$, again, a contradiction.

2. There are given N points in space (with $N > 1$). Some of them are connected to each other by lines. Prove that there are two points which are connected to the same number of points.

Solution: Any point is connected to 0, 1, ..., $N-1$ points. Assume that no two points are connected to the same number of points. Then, since there are N points, one (say p_0) must be connected to 0 points, one (say p_1) to 1 point, and so on, and the last point p_{N-1} is connected to $N-1$ points. However, that means that p_0 and p_{N-1} are connected, a contradiction.

3. Into how many regions do n lines divide the plane, assuming no two lines are parallel and no three lines intersect in the same point?

Solution: The answer is $\frac{n(n+1)}{2}+1$. We give two solutions.

#1. One observes that the number a_n of regions satisfies the recursive relation $a_n = a_{n-1}+n$, $n \geq 1$, since focusing on a given line l, we observe that it intersects exactly $n-1$ other lines, and hence itersects exactly n regions, splitting each one of them into two. Together with $a_0 = 1$, we have $a_n = a_0 + \sum_{j=1}^{n} j = \frac{n(n+1)}{2}+1$.

#2. One considers the numbers V, E, F of, respectively, vertices given by intersection points of the lines, edges into which the lines are subdivided by the intersection points, and regions in the complement of the lines. If we focus only on the numbers

V, E', F' of vertices, bounded edges, and bounded regions, Euler's formula for planar graphs implies $V - E' + F' = 1$. We observe that $-E + F = -E' + F'$, which gives $V - E + F = 1$. Now $V = \binom{n}{2} = \frac{n(n-1)}{2}$ since each two lines intersect at precisely one point, and $E = n^2$ since each line is subdivided into precisely n edges. Hence $F = 1 - V + E = 1 - \frac{n^2-n}{2} + n^2 = \frac{n(n+1)}{2} + 1$.

Note: both the relations $V - E' + F' = 1$ and $V - E + F = 1$ can be obtained by adding one unbounded face in the first case, and one vertex at infinity in the second case, and using Euler's formula

$$V - E + F = 2$$

for graphs in the (two-dimensional) sphere.

4. How many planes are needed to cut each of the edges of the cube at least twice (intersections at a vertex don't count as intersecting an edge)?

Solution: The answer is 4. The intersection of a plane with the cube is a polygon with at most 6 edges (as each edge must come from a face of the cube, of which there are 6), and thus with at most 6 vertices. To intersect each of the 12 edges of the cube twice, we would be forming at least 24 vertices, which correspond to at least 4 different intersection polygons. Thus at least 4 planes are needed.

The bound 4 can be realized:take the 4 standard regular hexagon sections of the cube.

5. Prove that there are no integers x, y and z (with $(x, y, z) \neq (0, 0, 0)$) such that

$$x^3 + 2y^3 + 4z^3 = 0.$$

Solution: Suppose that we have a solution x, y, z. Then we can assume that at least one of this numbers is odd, because if all of them are even, we divide all the numbers by 2 and obtain a new solution, and if all the numbers are even we can continue this process until we get a solution with at least one number odd.

Now, since $2y^3 + 4z^3$ is even, we conclude that x^3 (and thus x) is even. Thus $x = 2x_1$. We divide the equation by 2. Thus we get $4x_1^3 + y^3 + 2z^3 = 0$. Then y^3 must be even and $y = 2y_1$. Once again we divide the equation by 2 and get $2x_1^3 + 4y_1^3 + z^3 = 0$. Finally, z must be even, and thus we get a contradiction.

6. A certain country has finitely many cities. Any pair of these cities is connected by a road. However, all roads in this country are one-way roads, and it is therefore not always possible to travel from one city to another city. Show that the country has a city ("capital") that can be reached from every other city either directly or via exactly one intermediate city.

Solution: We give three solutions: first relying on mathematical induction, second relying on the technique of minimization, and third relying on both.

#1. We prove it by induction on n the number of cities. For two cities it is clear. The induction step from n to $n+1$ is shown as follows. Consider a city c' in our country C'. Let C be the sub-country of C' obtained by forgetting the city c' and all roads ending or starting at it. By induction hypothesis, C has a capital c. We claim that either c or c' is a capital for C. If the road between c and c' enters c, then c is a capital. Similarly, if there is a road from a city c_1, in C, to c, and the road between c' and c_1 (in C') enters c_1, then c is a capital. Hence if c is not a capital, then there is a road from c to c', and hence c' can be reached from c or from any city in C with that has a road to c, by a sequence of at most two roads. Moreover, for each sequence of two roads c_1 to c_2, c_2 to c (in C), which always exists as c is a capital of C, there must be a road from c_2 to c', and hence a sequence of two roads c_1 to c_2, c_2 to c'. Hence c' is a capital.

#2. (Inspired by solution of Julien Codsi) Let c be a city for which the number k of roads entering it is maximal among the n cities. We claim that c is a capital. Denote by $c_1, \ldots, c_k$ the cities from which there are roads entering c. For these cities the capital condition (i.e. being able to reach c by a sequence of at most two roads) is clearly satisfied. Now note that the number $l = n - k$ of roads exiting c is minimal among the n cities. Hence for each city d other than $c, c_1, \ldots, c_k$ there are at least l roads exiting it. However, since $l + k + 1 = n + 1 > n$, there must be at least one road that exits d and enters one of the cities $c, c_1, \ldots, c_k$, and therefore the capital condition is satisfied for d. This proves our claim.

#3. We prove it by induction on n the number of cities. For two cities it is clear. Denote the cities by $c_1, \ldots, c_n$, and for each c_i let A_i those cities c_j for which the road between c_i and c_j is from c_j to c_i, and B_i those where the road goes from c_i to c_j. Assume the statement is true for $n-1$. Pick a city c_i for which the size of A_i is minimal. Without loss of generality, assume it is c_n. The induction hypothesis gives a capital among $c_1, \ldots, c_{n-1}$, say c_{n-1}. If there is a road from c_n to c_{n-1}, there is nothing to prove. If not, the road goes from c_{n-1} to c_n and c_{n-1} belongs to A_n and c_n does NOT belong to A_{n-1}. Since A_n is of minimal size, there is an element of c_i of A_{n-1} which does not belong to A_n. That means that there is a road from c_n to c_i and a road from c_i to c_{n-1}. Thus c_{n-1} is the capital.

7. Three distinct points with integer coordinates lie in the plane on a circle of radius $r > 0$. Show that two points are separated by a distance of at least $r^{1/3}$.

Solution: First notice that any triangle (which is assumed above to be non-degenerate) with integer coordinates has area at least $\frac{1}{2}$. There are various ways to show this. For example, this is trivial if one of the sides is parallel to one of the axes, and then, for general triangles, one can show that it is always decompose them into such triangles. In another way, one observes that area S of the triangle with vertices $u, v, w \in \mathbb{R}^2$ is given by

$$S = \frac{1}{2}|\det(v-u, w-u)|,$$

where $v-u, w-u$ are considered as column vectors. Now if u, v, w have integer coordinates, so do $v-u, w-u$, and hence $2S$ is a positive integer, whence $2S \geq 1$.

On the other hand, the area of a triangle of sides a, b, and c which is inscribed into a circle of radius r is $\frac{abc}{4r}$. To see this, start from the formula $\frac{ab\sin\gamma}{2}$ where γ is the angle between a and b, and use that $\frac{c}{\sin\gamma} = 2r$. Then we have

$$\frac{abc}{4r} \geq \frac{1}{2}$$

If we assume that a, b, c are less or equal to $r^{1/3}$, we obtain $\frac{1}{4} \geq \frac{1}{2}$, a contradition.

Note: the fact that any triangle with integer coordinates has area at least $\frac{1}{2}$ is a particular case of Pick's theorem: a lattice polygon whose boundary consists of a sequence of connected nonintersecting straight-line segments has area $i + \frac{b}{2} - 1$, where i is the number of interior points, and b is the number of boundary points.

8. A convex pentagon has all its vertices lattice points in the plane (and no three collinear). Prove that its area is at least 5/2.

Solution: This is (Putnam '90, A3). We need Pick's theorem. So we have to show that a convex pentagon either has an interior lattice point or has at least two lattice points on its sides (apart from its vertices).

We can divide the coordinates into four parity classes: (odd, odd), (odd, even), (even, odd) and (even, even). At least two of the vertices must belong to the same class. But then their midpoint must be a lattice point. If the vertices are not adjacent, then the midpoint is an interior lattice point and we are done. So suppose the two vertices are A and B, adjacent and with midpoint M, also a lattice point. Let the other vertices be C, D, and E. A, C, D and E must all have different parity classes, otherwise we get another lattice point and we are done. But now M must have the same parity class as one of A, C, D and E, which gives us another lattice point.

Note that the result is the best possible. For example, (0, 0), (1, 0), (2, 2), (1, 2), (0, 1) has area exactly 5/2.