

**Concours Putnam**

Atelier de Pratique

Le jeudi, 15 octobre 14h30-15h30 (Salle: AA 1411)

**Games and Invariants**

1. Two players play the following game. Initially there is a pile of tokens. On each turn a player removes between 1 and 10 tokens. The player who is forced to take the last token loses.

Give a winning strategy for the first player if the game starts with 99 tokens, and a winning strategy for the second player if the game starts with 100 tokens.

2. Borgov places white bishops and Beth Harmon places black bishops on an  $8 \times 8$  chess-board, with Borgov playing first. A new bishop may be placed on an empty square only if it is not attacked by a bishop of the opposite color already on the board.

A player who cannot make a move loses. Give Beth Harmon a winning strategy.

3. We begin with the set of integers

$$\{1, 2, \dots, n\}.$$

At each step we choose two integers  $a \leq b$ , remove them, and replace them by  $b - a$ . We continue until only one integer remains.

Determine whether the final integer is odd or even in terms of  $n \pmod{4}$ .

4. There are 11 red chips, 30 white chips, and 19 blue chips. At each step we may replace two chips of two different colors by two chips of the third color. For example, we may replace one white chip and one blue chip by two red chips.

Can we ever reach a position in which two colors have the same number of chips?

5. Let  $s(n)$  denote the sum of the digits of  $n$  written in binary. Find all positive integers  $n$  such that

$$n + s(n) + s(s(n)) = 1000.$$

6. On a board are written the numbers

$$1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{100}.$$

At each step, choose two numbers  $a$  and  $b$ , erase them, and replace them by

$$a + b + ab.$$

Continue until only one number remains.

Find the final number and prove that it does not depend on the choices made.

7. The vertices of a finite graph are colored red or blue. Call a vertex *unstable* if more than half of its neighbors have the opposite color.

As long as there is an unstable vertex, choose one such vertex and change its color. Prove that after finitely many steps there will be no unstable vertices.

8. There are three piles of stones. Sisyphus repeatedly moves one stone from one pile to another.

Suppose that immediately before a move the source pile contains  $a$  stones and the destination pile contains  $b$  stones. Zeus pays Sisyphus

$$b - (a - 1) = b - a + 1$$

coins for this move; if this number is negative, Sisyphus pays Zeus the corresponding number of coins.

Eventually the three piles contain exactly the same numbers of stones as they did initially. What is the largest possible total profit of Sisyphus?

9. There are three piles containing 100, 101, and 102 stones. Alice and Bob alternately take one stone from one of the piles, with Alice moving first.

A player may not take a stone from the same pile from which that player took a stone on their previous turn. On a player's first turn there is no restriction.

A player who cannot make a move loses. Which player has a winning strategy?

10. We play the following game of number solitaire.

Start with a finite set  $S$  of distinct integers whose smallest element is 0 and whose largest element is  $n$ . If

$$m, m + 1 \in S \quad \text{but} \quad m + 2 \notin S,$$

we may remove  $m$  and  $m + 1$  from  $S$  and replace them by  $m + 2$ .

Prove that we can continue making moves until we obtain a set in which every two integers differ by at least 2, and that the largest element of the final set is either  $n$  or  $n + 1$ .