

Concours Putnam

Atelier de Pratique

Le jeudi, 8 octobre 14h30–15h30

Salle: Pavillon André-Aisenstadt 1411

Théorie des nombres

1. Find all solutions to

$$1! + 2! + 3! + \cdots + n! = m^2$$

in positive integers n and m .

Solution: When $k \geq 5$ the factorial $k!$ contains the factors 2 and 5, so it is divisible by 10; therefore, for every $n \geq 4$,

$$1! + 2! + \cdots + n! \equiv 1! + 2! + 3! + 4! = 33 \equiv 3 \pmod{10}.$$

On the other hand, a perfect square never ends in the digit 3 (the squares of $0, 1, \dots, 9$ end in $0, 1, 4, 9, 6, 5, 6, 9, 4, 1$). Thus there are no solutions with $n \geq 4$. It remains only to check $n \leq 3$: the sums are $1 = 1^2$, $1 + 2 = 3$, and $9 = 3^2$. The solutions are:

$$(n, m) = (1, 1) \quad \text{and} \quad (n, m) = (3, 3). \quad \square$$

2. Find all ordered pairs
- (a, b)
- of positive integers for which

$$\frac{1}{a} + \frac{1}{b} = \frac{3}{2018}.$$

Solution: Clearing denominators, the equation becomes $3ab - 2018a - 2018b = 0$. Multiplying by 3 and adding 2018^2 to both sides, it factors:

$$(3a - 2018)(3b - 2018) = 2018^2 = 2^2 \cdot 1009^2,$$

where 1009 is prime. Put $x = 3a - 2018$ and $y = 3b - 2018$, so $xy = 2018^2$ and $x, y > -2018$. Since $xy > 0$, the factors x and y have the same sign. If both were negative, we would have $|x|, |y| < 2018$ and so $xy < 2018^2$, which is impossible. Hence x and y are positive divisors of 2018^2 .

Since $2018 \equiv 2 \pmod{3}$, we need $x \equiv y \equiv -2018 \equiv 1 \pmod{3}$. Now $2 \equiv -1$ and $1009 \equiv 1 \pmod{3}$, so the divisor $x = 2^i 1009^j$ (with $0 \leq i, j \leq 2$) satisfies $x \equiv (-1)^i \pmod{3}$. Thus $i \in \{0, 2\}$, and then $y = 2^{2-i} 1009^{2-j} \equiv 1 \pmod{3}$ as well. Conversely, each such x makes a and b positive integers. This leaves the six values

$$x \in \{1, 4, 1009, 4 \cdot 1009, 1009^2, 4 \cdot 1009^2\},$$

and $a = (2018 + x)/3$, $b = (2018 + 2018^2/x)/3$ give the six solutions (a, b) :

$$(673, 1358114), \quad (674, 340033), \quad (1009, 2018), \\ (2018, 1009), \quad (340033, 674), \quad (1358114, 673).$$

□

3. Which members of the sequence

$$101, \quad 10101, \quad 1010101, \quad 101010101, \quad \dots$$

are prime?

Solution: Only 101. The member with $k \geq 2$ digits equal to 1 is

$$N_k = 1 + 10^2 + 10^4 + \dots + 10^{2(k-1)} = \frac{10^{2k} - 1}{99},$$

and $N_2 = 101$ is prime. Let $k \geq 3$.

If $k = 2\ell$ is even, then $\ell \geq 2$ and grouping the terms in pairs gives

$$N_k = (1 + 10^2)(1 + 10^4 + 10^8 + \dots + 10^{4(\ell-1)}) = 101 \cdot M, \quad M > 1,$$

so N_k is composite.

If k is odd, write $99N_k = (10^k - 1)(10^k + 1)$. Here $9 \mid 10^k - 1$, and $11 \mid 10^k + 1$ because $10 \equiv -1 \pmod{11}$ and k is odd. Hence

$$N_k = \frac{10^k - 1}{9} \cdot \frac{10^k + 1}{11},$$

a product of two integers, each larger than 1 since $k \geq 3$. So N_k is composite. □

4. Prove that the expression

$$\frac{\gcd(m, n)}{n} \binom{n}{m}$$

is an integer for all pairs of integers $n \geq m \geq 1$.

Solution: By Bézout's identity there are integers u and v with $\gcd(m, n) = um + vn$. Hence

$$\frac{\gcd(m, n)}{n} \binom{n}{m} = u \cdot \frac{m}{n} \binom{n}{m} + v \binom{n}{m}.$$

The second term is an integer. So is the first, because

$$\frac{m}{n} \binom{n}{m} = \frac{m}{n} \cdot \frac{n!}{m!(n-m)!} = \frac{(n-1)!}{(m-1)!(n-m)!} = \binom{n-1}{m-1}. \quad \square$$

5. Let f be a nonconstant polynomial with positive integer coefficients. Prove that if n is a positive integer, then $f(n)$ divides $f(f(n) + 1)$ if and only if $n = 1$.

Solution: For integers a and b , $a - b$ divides $a^i - b^i$ for every $i \geq 0$, so $a - b$ divides $f(a) - f(b)$ (this was used in problem 7 of the polynomial sheet). Taking $a = f(n) + 1$ and $b = 1$ gives

$$f(n) \mid f(f(n) + 1) - f(1),$$

so $f(n)$ divides $f(f(n) + 1)$ if and only if $f(n)$ divides $f(1)$.

If $n = 1$, then $f(1)$ certainly divides $f(1)$. If $n \geq 2$, then $f(1) < f(n)$, because f is nonconstant with positive coefficients and hence strictly increasing on $[0, \infty)$; since also $f(1) > 0$, the number $f(n)$ cannot divide $f(1)$. $\square$

6. Define a sequence $\{a_i\}$ by $a_1 = 3$ and $a_{i+1} = 3^{a_i}$ for $i \geq 1$. Which integers between 00 and 99 inclusive occur as the last two digits in the decimal expansion of infinitely many a_i ?

Solution: Only 87: in fact $a_i \equiv 87 \pmod{100}$ for every $i \geq 3$. We work our way up through the moduli 4, 20 and 100.

Modulo 4. Every a_i is odd, being a power of 3, and $3 \equiv -1 \pmod{4}$. Hence $a_{i+1} = 3^{a_i} \equiv (-1)^{a_i} = -1 \pmod{4}$, and also $a_1 = 3 \equiv -1$, so $a_i \equiv 3 \pmod{4}$ for all i .

Modulo 20. Since $3^4 = 81 \equiv 1 \pmod{20}$, the residue of 3^k modulo 20 depends only on k modulo 4. Therefore $a_{i+1} = 3^{a_i} \equiv 3^3 = 27 \equiv 7 \pmod{20}$, so $a_i \equiv 7 \pmod{20}$ for all $i \geq 2$.

Modulo 100. We have $3^5 = 243 \equiv 43$, $3^{10} \equiv 43^2 = 1849 \equiv 49$ and $3^{20} \equiv 49^2 = 2401 \equiv 1 \pmod{100}$. So the residue of 3^k modulo 100 depends only on k modulo 20, and for $i \geq 2$,

$$a_{i+1} = 3^{a_i} \equiv 3^7 = 2187 \equiv 87 \pmod{100}. \quad \square$$

7. Determine all positive integers n for which there exist positive integers a , b , and c satisfying

$$2a^n + 3b^n = 4c^n.$$

Solution: Only $n = 1$, for which $(a, b, c) = (1, 2, 2)$ works: $2 + 6 = 8$.

Suppose $n \geq 2$ and that a solution exists. The equation is homogeneous, so dividing a , b and c by their greatest common divisor gives another solution; we may therefore assume $\gcd(a, b, c) = 1$.

The case $n = 2$. Reducing $2a^2 + 3b^2 = 4c^2$ modulo 3 gives $-a^2 \equiv c^2$, that is $a^2 + c^2 \equiv 0 \pmod{3}$. A square is 0 or 1 modulo 3, so $a^2 \equiv c^2 \equiv 0$, that is, $3 \mid a$ and $3 \mid c$. Then 9 divides $4c^2 - 2a^2 = 3b^2$, so $3 \mid b^2$ and $3 \mid b$. This contradicts $\gcd(a, b, c) = 1$.

The case $n \geq 3$. First, $3b^n = 4c^n - 2a^n$ is even, so b is even. Then $2^n \mid b^n$, so $2a^n = 4c^n - 3b^n$ is divisible by 4, and a is even. Finally, $2a^n$ is divisible by 2^{n+1} and $3b^n$ by 2^n , so $4c^n$ is divisible by 2^n ; then $2^{n-2} \mid c^n$ with $n - 2 \geq 1$, and c is even. Again this contradicts $\gcd(a, b, c) = 1$. $\square$