

Concours Putnam

Atelier de Pratique

Le jeudi, 1 octobre 14h30-15h30 (Salle: Pavillon André-Aisenstadt 1411)

Fonctions et Integrals

1. Find all C^1 functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that for all x, y ,

$$f(x+y) - f(x-y) = 2yf'(x)$$

Solution: Putting $y = 1$, $f'(x) = \frac{1}{2}(f(x+1) - f(x-1))$. It follows that $f' \in C^1$, and so $f \in C^2$. Differentiating the original identity twice in y , we find

$$f''(x+y) = f''(x-y).$$

Thus f'' is symmetric around and x , and consequently f'' is constant. It follows that $f(x) = ax^2 + bx + c$ for some constants a, b, c , which is readily verified to satisfy the equation.

2. Let n be a positive integer, and define

$$f(n) = 1! + 2! + \dots + n!.$$

Find polynomials $P(x)$ and $Q(x)$ such that

$$f(n+2) = P(n)f(n+1) + Q(n)f(n)$$

for all $n \geq 1$.

Solution: This is (Putnam '84 B1) We have

$$f(n+2) - f(n+1) = (n+2)! = (n+2)(n+1)! = (n+2)(f(n+1) - f(n)),$$

hence

$$f(n+2) = (n+2)(f(n+1) - f(n)) + f(n+1) = (n+3)f(n+1) - (n+2)f(n),$$

and we can take $P(x) = x+3, Q(x) = -x-2$.

3. Find all functions $f : \mathbb{N} \rightarrow \mathbb{N}$ such that $f(nm) = f(n)f(m)$, and such that

$$\lim_{n \rightarrow \infty} \frac{\log(f(n))}{\log n} = 1$$

Solution: We will see that $f(n) = n$. Suppose the opposite. There is an m such that $\frac{f(m)}{m} = \alpha \neq 1$. Now we take the limit over the subsequence m^k with $k \rightarrow \infty$ and m fixed. Thus

$$1 = \lim_{k \rightarrow \infty} \frac{\log(f(m^k))}{\log(m^k)} = \frac{\log(\alpha m)}{\log m} = 1 + \frac{\log \alpha}{\log m}$$

and we get a contradiction if α is different from 1.

4. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function such that for every x ,

$$f(x+1) = \frac{1+f(x)}{1-f(x)}$$

Prove that f is periodic.

Solution: We have $f(x+2) = -1/f(x)$, hence $f(x+4) = f(x)$, and f is periodic with period 4.

5. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be continuous, with $f(x)f(f(x)) = 1$ for all $x \in \mathbb{R}$. If $f(1000) = 999$, find $f(500)$.

Solution: First notice that $f(1000)f(999) = 1$ implies that $f(999) = \frac{1}{999}$. Since $f(1000) = 999$, and the function is continuous, we conclude that there is a value a for which $f(a) = 500$. Then $f(500)f(a) = 1$ so that $f(500) = \frac{1}{f(a)} = \frac{1}{500}$.

6. Compute the following integrals

1. $\int_0^\infty \frac{x dx}{e^x - 1}$
2. $\int_0^1 \frac{\ln(1+x)}{x} dx$

Solution:

1.

$$\int_0^\infty \frac{x dx}{e^x - 1} = \int_0^\infty \frac{x e^{-x} dx}{1 - e^{-x}} = \int_0^\infty x e^{-x} \left(\sum_{k=0}^\infty e^{-kx} \right) dx.$$

Now use parts to see that for $\alpha < 0$, $\int_0^\infty x e^{\alpha x} dx = \frac{x e^{\alpha x}}{\alpha} \Big|_0^\infty - \int_0^\infty \frac{e^{\alpha x}}{\alpha} dx = \frac{1}{\alpha^2}$
 Thus we obtain

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} = \zeta(2).$$

2.

$$\begin{aligned} \int_0^1 \frac{\ln(1+x)}{x} dx &= \int_0^1 - \sum_{k=1}^{\infty} \frac{(-1)^k x^{k-1}}{k} dx = \sum_{k=1}^{\infty} (-1)^{k-1} \int_0^1 \frac{x^{k-1}}{k} dx \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2} \end{aligned}$$

Notice that

$$\begin{aligned} \zeta(2) + \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} &= \sum_{k=1}^{\infty} \frac{1}{k^2} + \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \\ &= 2 \sum_{k=1, \text{keven}}^{\infty} \frac{1}{k^2} = \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{2} \zeta(2) \end{aligned}$$

Then

$$\int_0^1 \frac{\ln(1+x)}{x} dx = \frac{\zeta(2)}{2}$$

7. Prove that $\int_0^1 \frac{dx}{x^x} = 1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \dots$

Solution:

$$\int_0^1 \frac{dx}{x^x} = \int_0^1 e^{-x \ln x} dx = \int_0^1 \sum_{k=0}^{\infty} \frac{(-1)^k x^k \ln^k x}{k!} dx$$

Let $I_k = \int_0^1 x^k \ln^k x dx$. Then

$$\begin{aligned} I_k &= \frac{x^{k+1}}{k+1} \ln^k x \Big|_0^1 - \int_0^1 \frac{k x^k}{k+1} \ln^{k-1} x dx \\ &= - \frac{k x^{k+1}}{(k+1)^2} \ln^{k-1} x \Big|_0^1 + \int_0^1 \frac{k(k-1) x^k}{(k+1)^2} \ln^{k-2} x dx \\ &\dots = (-1)^k \int_0^1 \frac{k! x^k}{(k+1)^k} dx = (-1)^k \frac{k!}{(k+1)^{k+1}}. \end{aligned}$$

Thus we get

$$\int_0^1 \frac{dx}{x^x} = \sum_{k=0}^{\infty} \frac{1}{(k+1)^{k+1}}$$

8. Prove or disprove the following statement: If F is a finite set with two or more elements, then there exists a binary operation $*$ on F such that for all x, y, z in F ,

1. $x * z = y * z$ implies $x = y$ (right cancellation holds), and
2. $x * (y * z) \neq (x * y) * z$ (no case of associativity holds).

Solution: This is (Putnam 84, B3) The statement is true. Let φ any bijection on F with no fixed points ($\varphi(x) \neq x$ for every x), and set $x * y = \varphi(x)$. Then

1. $x * z = y * z$ is equivalent to $\varphi(x) = \varphi(y)$, and this implies $x = y$ because φ is a bijection.
2. We have $x * (y * z) = \varphi(x)$ and $(x * y) * z = \varphi(\varphi(x))$, which cannot be equal because that would imply that $\varphi(x)$ is a fixed point of φ .

9. Let $f, g : [a, b] \rightarrow (0, \infty)$ be continuous, non-decreasing functions, such that $\int_a^b \sqrt{f} = \int_a^b \sqrt{g}$, and for all x we have

$$\int_a^x \sqrt{f} \leq \int_a^x \sqrt{g}.$$

Prove that

$$\int_a^b \sqrt{1+f} \geq \int_a^b \sqrt{1+g}.$$

Solution:

Let $\phi(x) = \int_a^x \sqrt{f}$, $\psi(x) = \int_a^x \sqrt{g}$. Then $\phi(a) = \psi(a) = 0$, $\phi(b) = \psi(b) =: c$, $\phi(x) \leq \psi(x)$.

Moreover by assumption ϕ', ψ' are non-decreasing, so ϕ, ψ are both convex functions.

We ought to show that

$$\int_a^b \sqrt{1+(\phi')^2} \geq \int_a^b \sqrt{1+(\psi')^2},$$

that is the the graph Γ_ϕ of ϕ is not shorter than the length of the graph of ψ .

Let Φ be the convex figure bounded by the graph of ϕ and the interval $J = [(a, 0), (b, c)]$. Similarly Ψ is the convex figure bounded by the graph of ψ and the same interval J .

Recall that if $K \subset L$ are two convex sets in the plane, then the perimeters satisfy $P(K) \leq P(L)$. This is easy to see from Crofton's formula: the length of a curve γ in the plane is given by a multiple of the integral of the integer-value function

$n_\gamma(\ell) = \#(\gamma \cap \ell)$ over the space of all lines (see Buffon's needle problem in wikipedia). If $K \subset L$ are both convex then any line ℓ intersects the boundary of K or L in either 0, 1 or 2 points (outside of a measure zero set of lines that might have an interval intersection with the boundary), and it's easy to see by convexity and inclusion that

$$n_{\partial K}(\ell) \leq n_{\partial L}(\ell)$$

for all lines.

By construction $\Psi \subset \Phi$, so

$$\text{Length}(\Gamma_\phi) + \text{Length}(J) \geq \text{Length}(\Gamma_\psi) + \text{Length}(J).$$

Cancelling out the second summand concludes the proof.