

Concours Putnam

Atelier de Pratique

Le jeudi, 1 octobre 14h30-15h30 (Salle: Pavillon André-Aisenstadt 1411)

Fonctions et Integrals

1. Find all C^1 functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that for all x, y ,

$$f(x+y) - f(x-y) = 2yf'(x)$$

2. Let n be a positive integer, and define

$$f(n) = 1! + 2! + \dots + n!.$$

Find polynomials $P(x)$ and $Q(x)$ such that

$$f(n+2) = P(n)f(n+1) + Q(n)f(n)$$

for all $n \geq 1$.

3. Find all functions $f : \mathbb{N} \rightarrow \mathbb{N}$ such that $f(nm) = f(n)f(m)$, and such that

$$\lim_{n \rightarrow \infty} \frac{\log(f(n))}{\log n} = 1$$

4. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function such that for every x ,

$$f(x+1) = \frac{1+f(x)}{1-f(x)}$$

Prove that f is periodic.

5. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be continuous, with $f(x)f(f(x)) = 1$ for all $x \in \mathbb{R}$. If $f(1000) = 999$, find $f(500)$.

6. Compute the following integrals

1. $\int_0^\infty \frac{xdx}{e^x-1}$
2. $\int_0^1 \frac{\ln(1+x)}{x} dx$

7. Prove that $\int_0^1 \frac{dx}{x^x} = 1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \dots$

8. Prove or disprove the following statement: If F is a finite set with two or more elements, then there exists a binary operation $*$ on F such that for all x, y, z in F ,

1. $x * z = y * z$ implies $x = y$ (right cancellation holds), and
2. $x * (y * z) \neq (x * y) * z$ (no case of associativity holds).

9. Let $f, g : [a, b] \rightarrow (0, \infty)$ be continuous, non-decreasing functions, such that $\int_a^b \sqrt{f} = \int_a^b \sqrt{g}$, and for all x we have

$$\int_a^x \sqrt{f} \leq \int_a^x \sqrt{g}.$$

Prove that

$$\int_a^b \sqrt{1+f} \geq \int_a^b \sqrt{1+g}.$$