Concours Putnam

Atelier de Pratique

Le jeudi, 24 septembre 14h30-15h30 (Salle: AA 1411)

Polynômes

1. Find a polynomial with integer coefficients whose zeros include

$$\sqrt{2}+\sqrt{5}.$$

Solution: Let

$$x=\sqrt{2}+\sqrt{5}.$$

Then

$$x^2=7+2\sqrt{10},$$

so

$$x^2-7=2\sqrt{10}.$$

Squaring once more gives

$$(x^2-7)^2=40.$$

Therefore

$$x^4-14x^2+49=40,$$

and hence

$$x^4-14x^2+9=0.$$

Thus one possible polynomial is

$$P(x)=x^4-14x^2+9.$$

2. Let $P(x)$ be a polynomial of degree n. Suppose that

$$P(0),P(1),\ldots,P(n)$$

are all integers. Prove that $P(m)$ is an integer for every integer m.

Solution: Define the forward difference operator by

$$(\Delta f)(x)=f(x+1)-f(x).$$

Since $P(0),\ldots,P(n)$ are integers, all the numbers

$$P(0),\quad \Delta P(0),\quad \Delta^2 P(0),\quad \ldots,\quad \Delta^n P(0)$$

are integers. Newton's interpolation formula gives

$$P(x) = \sum_{j=0}^{n} \Delta^j P(0) \binom{x}{j}, \qquad \binom{x}{j} = \frac{x(x-1)\cdots(x-j+1)}{j!}.$$

For every integer m, each generalized binomial coefficient $\binom{m}{j}$ is an integer. For $m \geq 0$ this is the usual binomial coefficient, while for $m < 0$ one may use

$$\binom{m}{j} = (-1)^j \binom{-m+j-1}{j}.$$

Hence every term in the sum is an integer, and therefore $P(m) \in \mathbb{Z}$. $\square$

3. The product of two of the four zeros of the quartic equation

$$x^4 - 18x^3 + kx^2 + 200x - 1984 = 0$$

is -32. Find k.

Solution: Let the two roots whose product is -32 have sum a, and let the other two roots have sum b. Since the product of all four roots is -1984, the product of the second pair is

$$\frac{-1984}{-32} = 62.$$

Thus the polynomial factors as

$$(x^2 - ax - 32)(x^2 - bx + 62).$$

Comparing the coefficient of x^3 gives $a + b = 18$, while comparing the coefficient of x gives

$$-62a + 32b = 200.$$

Substituting $b = 18 - a$ yields $a = 4$ and $b = 14$. Therefore the coefficient of x^2 is

$$k = ab + 62 - 32 = 4 \cdot 14 + 30 = 86.$$

Hence $k = 86$. $\square$

4. Let a, b, c be distinct integers. Can the polynomial

$$(x-a)(x-b)(x-c) - 1$$

be factored into the product of two nonconstant polynomials with integer coefficients?

Solution: No. A reducible monic cubic polynomial over $\mathbb{Z}$ must have an integer root r. If r were a root, then

$$(r-a)(r-b)(r-c) = 1.$$

The three factors are integers, so each of them must be ± 1. But $r-a$, $r-b$, and $r-c$ are three distinct integers, because a, b, c are distinct, whereas there are only two integers in $\{-1, 1\}$. This is impossible. Hence the polynomial cannot be factored in the required way. $\square$

5. Let $P(x) \in \mathbb{Z}[x]$ have degree $n > 5$ and n distinct integer roots. Prove that $P(x) + 3$ also has n distinct real roots.

Solution: Write the roots in increasing order,

$$a_1 < a_2 < \cdots < a_n,$$

so that

$$P(x) = A \prod_{j=1}^{n} (x - a_j), \qquad A \in \mathbb{Z}, \quad A \neq 0.$$

For each $i = 1, \ldots, n-1$, put $x_i = a_i + \frac{1}{2}$. Since the roots are distinct integers, $x_i \in (a_i, a_{i+1})$. Among the factors $|x_i - a_j|$, the six smallest possible ones are at least

$$\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{3}{2}, \frac{5}{2}, \frac{5}{2}.$$

All remaining factors, if any, have absolute value greater than 1. Therefore

$$|P(x_i)| \geq |A| \left(\frac{1}{2}\right)^2 \left(\frac{3}{2}\right)^2 \left(\frac{5}{2}\right)^2 = |A| \frac{225}{64} > 3.$$

The sign of P is constant on each interval between consecutive roots and alternates from one such interval to the next. Thus, on every bounded interval (a_i, a_{i+1}) where $P < 0$, the graph starts at 0, takes a value below -3, and returns to 0. By the intermediate value theorem, $P(x) = -3$ has at least two roots in every such interval. On each unbounded interval where $P < 0$, it has at least one root.

Counting the negative intervals gives at least n distinct solutions of $P(x) = -3$: if n is even there are either $n/2$ bounded negative intervals, or $n/2 - 1$ bounded negative intervals together with both unbounded ones; if n is odd there are $(n-1)/2$ bounded negative intervals together with one unbounded one. Since $P(x) + 3$ has degree n, it cannot have more than n real roots. Hence it has exactly n distinct real roots. $\square$

6. Let $f(x)$ be a polynomial with real coefficients, and suppose that

$$f(x) + f'(x) > 0$$

for all real x. Prove that $f(x) > 0$ for all real x.

Solution: Consider

$$g(x) = \mathrm{e}^x f(x).$$

Then

$$g'(x) = \mathrm{e}^x (f(x) + f'(x)) > 0,$$

so g is strictly increasing. Since f is a polynomial,

$$\lim_{x \to -\infty} \mathrm{e}^x f(x) = 0.$$

Thus, for every real x,

$$g(x) > \lim_{t \to -\infty} g(t) = 0.$$

Since $\mathrm{e}^x > 0$, it follows that $f(x) > 0$. $\square$

7. Let $k \geq 6$ and let $P(x) \in \mathbb{Z}[x]$. Suppose that at k distinct integer arguments, all the values of P belong to

$$\{1, 2, \ldots, k-1\}.$$

Prove that all these k values are equal.

Solution: Let the k integer arguments be

$$x_1 < x_2 < \cdots < x_k.$$

For integers u, v, the difference $P(u) - P(v)$ is divisible by $u - v$. Hence

$$x_k - x_1 \mid P(x_k) - P(x_1).$$

But $x_k - x_1 \geq k - 1$, while

$$|P(x_k) - P(x_1)| \leq k - 2.$$

Therefore $P(x_k) = P(x_1)$; denote this common value by c. It follows that

$$P(x) - c = (x - x_1)(x - x_k)Q(x)$$

for some $Q(x) \in \mathbb{Z}[x]$.

Now let $3 \leq i \leq k-2$. If $P(x_i) \neq c$, then $Q(x_i)$ is a nonzero integer, and hence

$$|P(x_i) - c| \geq |(x_i - x_1)(x_i - x_k)| \geq 2(k-3) > k-2,$$

contradicting the fact that both $P(x_i)$ and c lie in $\{1, \dots, k-1\}$. Thus

$$P(x_3) = \cdots = P(x_{k-2}) = c.$$

Consequently,

$$P(x) - c = (x - x_1)(x - x_3) \cdots (x - x_{k-2})(x - x_k)R(x)$$

with $R(x) \in \mathbb{Z}[x]$.
If $P(x_2) \neq c$, then $R(x_2)$ is a nonzero integer, so

$$|P(x_2) - c| \geq (k-4)!(k-2) > k-2,$$

again impossible. Hence $P(x_2) = c$. The same argument at x_{k-1} gives $P(x_{k-1}) = c$. Therefore all k values are equal. $\square$

8. Two players A and B play the following game. A thinks of a polynomial with nonnegative integer coefficients. B must guess the polynomial. B has two shots: she can pick a number and ask A to return the polynomial value there, and then she has one more such try. Can B win the game?

Solution: Yes. For the first query, B asks for $P(1)$. Suppose

$$P(1) = S.$$

If $S = 0$, then all coefficients are zero, so $P = 0$ and B is already done. Otherwise, every coefficient of P is an integer between 0 and S. For the second query, B asks for $P(S+1)$. Writing

$$P(x) = a_0 + a_1 x + \cdots + a_d x^d,$$

we obtain

$$P(S+1) = a_0 + a_1(S+1) + \cdots + a_d(S+1)^d.$$

Because $0 \leq a_i \leq S$, this is precisely the base-$(S+1)$ expansion of the integer $P(S+1)$, whose digits are the coefficients $a_0, a_1, \dots, a_d$. Thus the second answer determines all coefficients of P, and B can always win. $\square$

9. If $a, b, c > 0$, is it possible that each of the polynomials

$$P(x) = ax^2 + bx + c, \qquad Q(x) = cx^2 + ax + b, \qquad R(x) = bx^2 + cx + a$$

has two real roots?

Solution: No. If all three quadratics had two distinct real roots, their discriminants would all be positive:

$$b^2 > 4ac, \qquad a^2 > 4bc, \qquad c^2 > 4ab.$$

Multiplying these inequalities gives

$$a^2b^2c^2 > 64a^2b^2c^2,$$

which is impossible because $abc > 0$. Thus the three polynomials cannot all have two real roots. $\square$