# Concours Putnam

Atelier de Pratique
Le jeudi, 17 septembre 14h30–15h30
Salle: Pavillon André-Aisenstadt 1411

**Le principe des tiroirs (Dirichlet)**

1. Five points are chosen in a closed equilateral triangle of side 1. Show that two of them are at distance at most $1/2$.

**Solution:** Joining the midpoints of the three sides cuts the triangle $T$ into four closed equilateral triangles of side $1/2$, whose union is all of $T$. Since there are five points and four triangles, the pigeonhole principle guarantees one of the small triangles contains two or more of the points.

It remains to show that two points of an equilateral triangle of side $1/2$ are at distance at most $1/2$. In general the diameter of a convex polygon is the length of its longest side; this can be shown by optimization of the convex function $x \mapsto |x - p|$ for any fixed point $p$ (one optimizes with respect to $x$ and then with respect to $p$). $\square$

2. Let $n \geq 1$ and let $a_1, a_2, \ldots, a_n$ be integers. Show that some block of consecutive terms has sum divisible by $n$: there are $i \leq j$ with

$$n \mid a_i + a_{i+1} + \cdots + a_j.$$

**Solution:** Consider the $n+1$ partial sums:

$$S_0 = 0, \qquad S_k = a_1 + a_2 + \cdots + a_k \quad (1 \leq k \leq n).$$

There are only $n$ residues modulo $n$, so two of the $S_k$ are congruent: $S_i \equiv S_j \pmod{n}$ with $0 \leq i < j \leq n$. Then

$$a_{i+1} + a_{i+2} + \cdots + a_j = S_j - S_i \equiv 0 \pmod{n}.$$

This block is nonempty since $i < j$. $\square$

3. Let $n \geq 1$ and let $A \subseteq \{1, 2, \ldots, 2n\}$ be a set of $n+1$ integers. Show that
   (a) two elements of $A$ are relatively prime;
   (b) one element of $A$ divides another.

Show that both statements fail for sets of $n$ elements.

**Solution:** (a) Split $\{1, \ldots, 2n\}$ into the $n$ pairs $\{1, 2\}, \{3, 4\}, \ldots, \{2n-1, 2n\}$. Since $|A| = n + 1$, two elements of $A$ lie in the same pair, i.e., $A$ contains two consecutive integers $k$ and $k + 1$. Clearly $\gcd(k, k + 1) = 1$.

(b) Write each $a \in A$ as $a = 2^e m$ with $m$ odd; then $m \in \{1, 3, 5, \ldots, 2n - 1\}$, which has $n$ elements. Since $|A| = n + 1$, two elements of $A$ have the same odd part: $a = 2^e m$ and $b = 2^f m$ with $e < f$. Then $a \mid b$. $\square$

Both fail with $n$ elements: $A = \{2, 4, \ldots, 2n\}$ has no two coprime elements, and $A = \{n + 1, n + 2, \ldots, 2n\}$ has no element dividing another, because twice the smallest element already exceeds $2n$.

*Remark.* Part (b) is an observation of Erdős.

4. Six people meet, and any two of them either know each other or do not. Show that there are three who know each other pairwise, or three who are pairwise strangers. Show that this is false for five people.

**Solution:** Let $G$ be the graph on the six people whose edges join pairs who know each other; we must find a triangle in $G$ or in its complement $\overline{G}$.

Fix a person $v$. The five others are partitioned into the acquaintances of $v$ and the strangers to $v$, so, by the pigeon hole principle, one of these two sets has at least three elements. Suppose first that $a, b, c$ all know $v$. If two of $a, b, c$ know each other, then we are done; if not, then $a, b, c$ are pairwise strangers, and we are done again. The other case is the same argument with the roles of "knows" and "is a stranger to" reversed.

For five people the conclusion fails: place people at the vertices of a pentagon and let two people know each other if and only if they are adjacent. The resulting graph is a 5-cycle, which has no triangle, and its complement is the pentagram, which is again a 5-cycle and so has no triangle either. $\square$

*Remark.* This is a problem from the 1953 version of the Putnam exam, which was stated as: if the 15 edges of $K_6$ are coloured red or blue, prove there exists a monochromatic triangle.

5. Show that from any ten distinct two-digit integers one can select two disjoint nonempty subsets whose elements have the same sum.

**Solution:** Let $A$ be the given set of ten integers, so $A \subseteq \{10, 11, \ldots, 99\}$. It has $2^{10} - 1 = 1023$ nonempty subsets. The sum of the elements of such a subset is at least 10 and at most

$$90 + 91 + \cdots + 99 = 945,$$

so it takes at most $\#([10, 945] \cap \mathbb{Z}) = 936$ values. Because $1023 > 936$, two distinct nonempty subsets $B \neq C$ have the same sum (by the pigeon hole principle!).

The sets $B, C$ need not be disjoint, but there is a trick: let $B' = B \setminus C$ and $C'C \setminus B$. Deleting the elements of $B \cap C$ *removes the same amount from each sum*, so the two sums of $B'$ and $C'$ are still equal, and $B' \cap C' = \emptyset$ two sets are now disjoint. $\square$

6. Let $n \geq 1$ and let $a_1, a_2, \ldots, a_{n^2+1}$ be distinct real numbers. Show that some subsequence of $n + 1$ terms is increasing, or some subsequence of $n + 1$ terms is decreasing. Show that $n^2 + 1$ cannot be replaced by $n^2$.

**Solution:** For each index $i$ let

$$\begin{aligned} x_i &= \text{the length of the longest increasing subsequence ending at } a_i, \\ y_i &= \text{the length of the longest decreasing subsequence ending at } a_i, \end{aligned}$$

so $x_i, y_i \geq 1$. Suppose, for contradiction, that there is no monotone subsequence of length $n + 1$. Then $x_i, y_i \in \{1, 2, \ldots, n\}$ for every $i$, so the map

$$i \longmapsto (x_i, y_i)$$

sends the $n^2 + 1$ indices into a set of only $n^2$ pairs. By the pigeonhole principle there are $i < j$ with $(x_i, y_i) = (x_j, y_j)$.

If $a_i < a_j$, appending $a_j$ to a longest increasing subsequence ending at $a_i$ produces an increasing subsequence of length $x_i + 1$ ending at $a_j$, so $x_j \geq x_i + 1$. If $a_i > a_j$, the same argument gives $y_j \geq y_i + 1$. Either way $(x_i, y_i) \neq (x_j, y_j)$, which is the desired contradiction. $\square$

We now explain why the bound is sharp; the sequence of $n^2$ numbers:

$$\underbrace{n, n-1, \ldots, 1}_{\text{block 1}}, \quad \underbrace{2n, 2n-1, \ldots, n+1}_{\text{block 2}}, \quad \ldots, \quad \underbrace{n^2, n^2-1, \ldots, n^2-n+1}_{\text{block } n}$$

has no monotone subsequence of length $n + 1$, because an increasing subsequence uses at most one term per block, and a decreasing one stays inside a single block.

*Remark.* This is the Erdős–Szekeres theorem (1935), from the paper in which they proved that any sufficiently large set of points in general position contains the vertices of a convex $n$-gon.

7. For $t \in \mathbb{R}$ write $\{t\} = t - \lfloor t \rfloor \in [0, 1)$ for its fractional part.

   (a) *(Dirichlet)* Let $\alpha \in \mathbb{R}$ and let $N$ be a positive integer. Show that there are integers $p$ and $q$ with $1 \le q \le N$ and
   $$|q\alpha - p| < \tfrac{1}{N}.$$

   (b) Deduce that if $\alpha$ is irrational then $\mathbb{Z} + \alpha\mathbb{Z} = \{a + b\alpha : a, b \in \mathbb{Z}\}$ is dense in $\mathbb{R}$. In particular $\mathbb{Z} + \sqrt{2}\,\mathbb{Z}$ is dense.

**Solution:** (a) Consider the $N + 1$ numbers
$$\{0\},\ \{\alpha\},\ \ldots,\ \{N\alpha\}\ \in [0, 1)$$
and the $N$ intervals $[k/N, (k + 1)/N)$, $k = 0, 1, \ldots, N - 1$, which partition $[0, 1)$. By the pigeonhole principle two of the numbers lie in the same interval, say $\{i\alpha\}$ and $\{j\alpha\}$ with $0 \le i < j \le N$; then $|\{j\alpha\} - \{i\alpha\}| < 1/N$. Put:
$$q = j - i \in \{1, \ldots, N\}, \qquad p = \lfloor j\alpha \rfloor - \lfloor i\alpha \rfloor.$$
Then $q\alpha - p = \{j\alpha\} - \{i\alpha\}$, so $|q\alpha - p| < 1/N$ as required.

(b) Let $x \in \mathbb{R}$ and $\varepsilon > 0$. Choose $N$ with $1/N < \varepsilon$ and take $p, q$ as in (a). Set
$$\delta = q\alpha - p \in \mathbb{Z} + \alpha\mathbb{Z}, \qquad |\delta| < \tfrac{1}{N} < \varepsilon.$$
Moreover $\delta \ne 0$ (otherwise $\alpha = p/q$ would be rational). The multiples $m\delta$, $m \in \mathbb{Z}$ belong to $\mathbb{Z} + \alpha\mathbb{Z}$, and they form an arithmetic progression of step $|\delta| < \varepsilon$ running through all of $\mathbb{R}$. Hence some $m\delta$ lies within $\varepsilon$ of $x$. □

*Remark.* Part (a) is Dirichlet's approximation theorem (1842); it states that every irrational $\alpha$ admits infinitely many rational approximations $p/q$ satisfying:
$$|\alpha - p/q| < 1/q^2,$$
which one proves by dividing the above by $q$ and letting $N \to \infty$.