Concours Putnam

Atelier de Pratique
Le jeudi, 17 septembre 14h30–15h30
Salle: Pavillon André-Aisenstadt 5448

Le principe des tiroirs (Dirichlet)

1. Five points are chosen in a closed equilateral triangle of side 1. Show that two of them are at distance at most 1/2.

2. Let $n \geq 1$ and let $a_1, a_2, \ldots, a_n$ be integers. Show that some block of consecutive terms has sum divisible by n: there are $i \leq j$ with
$$n \mid a_i + a_{i+1} + \cdots + a_j.$$

3. Let $n \geq 1$ and let $A \subseteq \{1, 2, \ldots, 2n\}$ be a set of $n+1$ integers. Show that
   (a) two elements of A are relatively prime;
   (b) one element of A divides another.

   Show that both statements fail for sets of n elements.

4. Six people meet, and any two of them either know each other or do not. Show that there are three who know each other pairwise, or three who are pairwise strangers. Show that this is false for five people.

5. Show that from any ten distinct two-digit integers one can select two disjoint nonempty subsets whose elements have the same sum.

6. Let $n \geq 1$ and let $a_1, a_2, \ldots, a_{n^2+1}$ be distinct real numbers. Show that some subsequence of $n+1$ terms is increasing, or some subsequence of $n+1$ terms is decreasing. Show that n^2+1 cannot be replaced by n^2.

7. For $t \in \mathbb{R}$ write $\{t\} = t - \lfloor t \rfloor \in [0, 1)$ for its fractional part.
   (a) *(Dirichlet)* Let $\alpha \in \mathbb{R}$ and let N be a positive integer. Show that there are integers p and q with $1 \leq q \leq N$ and
   $$|q\alpha - p| < \tfrac{1}{N}.$$
   (b) Deduce that if α is irrational then $\mathbb{Z} + \alpha\mathbb{Z} = \{a + b\alpha : a, b \in \mathbb{Z}\}$ is dense in $\mathbb{R}$. In particular $\mathbb{Z} + \sqrt{2}\,\mathbb{Z}$ is dense.