Concours Putnam

Atelier de Pratique

Le jeudi, 26 novembre 14h30-15h30 (Salle: AA 1411)

Mixed Problems

1. In a round-robin tournament with n players, every pair of players plays exactly one match, and there are no ties. Let W_k and L_k be the numbers of wins and losses of player k. Prove that
$$\sum_{k=1}^{n} W_k^2 = \sum_{k=1}^{n} L_k^2.$$

2. Prove that no integer $n > 1$ divides $2^n - 1$.

3. Let $(a_n)_{n\geq 1}$ be a sequence of positive real numbers such that
$$a_n \leq a_{2n} + a_{2n+1} \qquad (n \geq 1).$$
Prove that $\sum_{n=1}^{\infty} a_n$ diverges.

4. Let A, B, C, D be four distinct points in the plane, no three collinear. Suppose that AB^2, AC^2, AD^2, BC^2, BD^2, and CD^2 are all rational numbers. Prove that
$$\frac{\text{area}(\triangle ABC)}{\text{area}(\triangle ABD)}$$
is rational.

5. Evaluate the improper integral
$$\int_0^{\infty} \frac{\arctan(\pi x) - \arctan x}{x}\, dx.$$

6. A sequence $a_1, \ldots, a_{50}$ of real numbers is called *p-balanced* if the sums
$$a_r + a_{r+p} + a_{r+2p} + \cdots$$
are equal for $r = 1, \ldots, p$, with each sum containing only terms whose indices are at most 50. Suppose the sequence is p-balanced for each of $p = 3, 5, 7, 11, 13, 17$. Prove that $a_1 = \cdots = a_{50} = 0$.

7. Let $n \geq 1$ and let
$$P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + 1$$
be a polynomial with nonnegative real coefficients, all of whose roots are real. Prove that $P(2) \geq 3^n$.

8. A partition of a set is a collection of disjoint nonempty parts whose union is the set. For a partition π of $\{1, 2, \ldots, 9\}$, let $\pi(x)$ denote the number of elements in the part containing x. Prove that for any two partitions π and π', there are distinct $x, y \in \{1, 2, \ldots, 9\}$ such that
$$\pi(x) = \pi(y) \qquad \text{and} \qquad \pi'(x) = \pi'(y).$$