Concours Putnam

Atelier de Pratique
Le jeudi, 19 novembre 14h30-15h30
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Suites et Séries

1. Compute the following

   1. $\sum_{n=0}^{\infty} \frac{n}{2^n}$
   2. $\sum_{n=1}^{\infty} \frac{1}{n2^n}$
   3. $\sum_{n=1}^{\infty} \frac{1}{n(n+3)}$
   4. $\sum_{n=0}^{\infty} \frac{(n+1)^2}{n!}$
   5. $\sum_{n=0}^{\infty} \binom{n+k}{k} x^n$ for $k = 0, 1, 2, ... |x| < 1$.

Solution:

1. 2. We have $\sum_{n=0}^{\infty} \frac{n}{2^n} = \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} \frac{1}{2^n} = \sum_{k=1}^{\infty} \frac{2}{2^k} = 2$.
2. $\log 2$. Put $x = -\frac{1}{2}$ in the logarithmic series.
3. $\frac{11}{18} = \frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{3}\right)$. Write $\frac{1}{n(n+1)} = \frac{1}{3}\left(\frac{1}{n} - \frac{1}{n+3}\right)$.
4. $5e$. Write $(n+1)^2 = n(n-1) + 3n + 1$ and use the exponential series.
5. $\frac{1}{(1-x)^{k+1}}$. Write $\sum_{n=0}^{\infty} \binom{n+k}{k} x^n = \sum_{n=0}^{\infty} \frac{(-k-1)(-k-2)...(-k-n)}{n!}(-x)^n = (1-x)^{-k-1}$.

2. Evaluate

$$\sum_{n=1}^{\infty} \frac{(-1)^{[2^n x]}}{2^n} \quad 0 < x < 1,$$

where $[t]$ denotes the greatest integer $\leq t$.

Solution: The numerator of the n^{th} term is $1 - 2k$ where $k \in \{0, 1\}$ is the n^{th} digit in the binary expansion of x. The answer is, then, $1 - 2x$.

3. Evaluate

$$\sum_{n=1}^{\infty} \frac{s(n)}{n(n+1)},$$

where $s(n)$ is the number of 1's in the binary expansion of n.

Solution: This is (Putnam '81, B5). Let $S = \sum_{n=1}^{\infty} \frac{s(n)}{n(n+1)}$. Since $s(2n) = s(n)$ and $s(2n+1) = s(2n)+1$, it follows that

$$S - \frac{s(1)}{2} = \sum_{n=1}^{\infty} \frac{s(2n)}{2n(2n+1)} + \frac{s(2n+1)}{(2n+1)(2n+2)} = \sum_{n=1}^{\infty} \frac{s(n)}{2n(n+1)} + \frac{1}{(2n+1)(2n+2)}.$$

Thus, $S - \frac{1}{2} = \frac{S}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$. Then $S = 2\log 2$.

4. Let a and d be positive integers. Show that the arithmetic progression a, $a+d$, $a+2d, \ldots$ either contains no perfect square or contains infinitely many perfect squares.

Solution: Let n^2 be an element of the sequence. Then $(n+kd)^2 = n^2 + (2kn+k^2d)d$ is also an element of the sequence, for any positive integer k.

5. Solve: $x_{n+1} = 2x_n(1-x_n)$, with $x_1 = -1$.

Solution: We write $x_n = \frac{1-y_n}{2}$. Thus the equation becomes

$$\frac{1-y_{n+1}}{2} = (1-y_n)\frac{1+y_n}{2}$$

$$y_{n+1} = y_n^2$$

Thus $y_n = 3^{2^{n-1}}$ and $x_n = \frac{1-3^{2^{n-1}}}{2}$.

6. Let $\{x_n\}$ be a sequence of real numbers satisfying $x_n = (x_{n-1} + x_{n-2})/2$. Show that the sequence converges, and find the limit in terms of x_0 and x_1.

Solution: The characteristic equation is given by

$$2x^2 = x + 1$$

The roots are 1 and $-\frac{1}{2}$. Thus, $x_n = \frac{x_0+2x_1}{3} + \left(-\frac{1}{2}\right)^n \frac{2(x_0-x_1)}{3}$. The limit is clearly $\frac{x_0+2x_1}{3}$.

7. Let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be infinite sequences of positive integers. Show that there exist distinct indices p and q such that $x_p \geq x_q$, $y_p \geq y_q$, and $z_p \geq z_q$.

Solution: First, we claim that there is a (fixed) index q for which there are infinitely many p such that $x_p \geq x_q$ and $y_p \geq y_q$. First take and x_q such that the sequence $\{x_n\}$ reaches a minimal value in x_q. Then all the other elements in the sequence satisfy $x_p \geq x_q$. Now we look at y_q. If there are infinitely many y_p such that $y_p \geq y_q$, then the claim is true. If not, then there are infinitely many elements in the sequence $\{y_n\}$ taking finitely many values between 1 and $y_q - 1$. Therefore, there are infinitely many indexes that take the same value. We take such a subsequence and look at the situation in $\{x_n\}$. There is a minimal element there and we rename that element x_q. This proves the claim.

Now we look at z_q. If there is a $z_p \geq z_q$, we are done. If not, we proceed as before and take ininitely many indexes in the sequence $\{z_n\}$ that take the same value. Now we take those subsequences in $\{x_n\}$ and $\{y_n\}$ and apply the previous claim.

8. Let $a_1 = a_2 = 1$ and $a_n = \frac{a_{n-1}^2+2}{a_{n-2}}$ for $n \geq 3$. Show that a_n is an integer for all n.

Solution: We write $a_n a_{n-2} - a_{n-1}^2 = 2 = a_{n-1}a_{n-3} - a_{n-2}^2$. Thus

$$\frac{a_n + a_{n-2}}{a_{n-1}} = \frac{a_{n-1} + a_{n-3}}{a_{n-2}},$$

and therefore, $b_n = \frac{a_n+a_{n-2}}{a_{n-1}}$ is a constant sequence. Placing $n = 3$ we obtain $b_n = 4$. Thus, $a_n = 4a_{n-1} - a_{n-2}$ and since a_1, a_2 are integers, the same is true for a_n.

9. For each integer $n \geq 0$, let $d(n) = n - m^2$, where m is the largest integer with $m^2 \leq n$. Define a sequence $\{b_k\}$ by $b_0 = B$; $b_{k+1} = b_k + d(b_k)$. For what positive integers B is $\{b_k\}$ eventually constant?

Solution: This is (Putnam '91, B1). If b_k is a square, then $d(b_k) = 0$, so $b_{k+1} = b_k$ and the sequence is constant from that point on.

If b_k is not a square, then for some m it lies between m^2 and $(m+1)^2$, so we can write it as $m^2 + r$, where $1 \leq r \leq 2m$. So $b_{k+1} = m^2 + 2r$. But $m^2 < b_{k+1} < (m+2)^2$ and $b_{k+1} \neq (m+1)^2$, so b_{k+1} is not a square (and is $> b_k$). Thus $\{b_k\}$ is eventually constant iff B is a square.

10. A sequence of natural numbers is given by $x_1 = 2$, $x_{n+1} = \lfloor 1.5x_n \rfloor$. Prove that it contains infinitely many odd numbers, and infinitely many even numbers

Solution: Say $x_k = 2^a b$, where b is odd. Then $x_{k+a} = 3^a b$ is odd. Thus there are infinitely many odd elements in the sequence.

Assume now the sequence is odd from a certain place, say from x_k. Then setting $y_n = x_{k+n}$, it holds that $x_{k+n} - 1 = (3/2)^n (x_k - 1)$. Putting $x_k = 2^a b + 1$ with b odd, we see that $x_{k+a} = 3^a b + 1$ which is even, a contradiction.