Concours Putnam

Atelier de Pratique

Le mardi, 7 octobre 13h30-14h30

Des jeus et des invariants

1. Let $s(n)$ be the sum of the digits of n written in base 10. Find all integers n for which $n + s(n) + s(s(n)) = 1000$.

Solution: Since $s(n) \equiv n \pmod 3$ we would have $100 \equiv n + n + s(n) \equiv 3n \equiv 0 \pmod 3$ which is impossible.

2. Given that $\{x_1, x_2, \ldots, x_n\} = \{1, 2, \ldots, n\}$, find, with proof, the largest possible value, as a function of n (with $n \geq 2$), of

$$x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1.$$

Solution: This Putnam 1996 B-3.

View $x_1, \ldots, x_n$ as an arrangement of the numbers $1, 2, \ldots, n$ on a circle. We prove that the optimal arrangement is

$$\ldots, n-4, n-2, n, n-1, n-3, \ldots$$

To show this, note that if a, b is a pair of adjacent numbers and c, d is another pair (read in the same order around the circle) with $a < d$ and $b > c$, then the segment from b to c can be reversed, increasing the sum by

$$ac + bd - ab - cd = (d-a)(b-c) > 0.$$

Now relabel the numbers so they appear in order as follows:

$$\ldots, a_{n-4}, a_{n-2}, a_n = n, a_{n-1}, a_{n-3}, \ldots$$

where without loss of generality we assume $a_{n-1} > a_{n-2}$. By considering the pairs a_{n-2}, a_n and a_{n-1}, a_{n-3} and using the trivial fact $a_n > a_{n-1}$, we deduce $a_{n-2} > a_{n-3}$. We then compare the pairs a_{n-4}, a_{n-2} and a_{n-1}, a_{n-3}, and using that $a_{n-1} > a_{n-2}$, we deduce $a_{n-3} > a_{n-4}$. Continuing in this fashion, we prove that $a_n > a_{n-1} > \ldots > a_1$ and so $a_k = k$ for $k = 1, 2, \ldots, n$, i.e. that the optimal arrangement is as claimed. In particular, the maximum value of the sum is

$$\begin{aligned}
&1 \cdot 2 + (n-1) \cdot n + 1 \cdot 3 + 2 \cdot 4 + \cdots + (n-2) \cdot n \\
&= 2 + n^2 - n + (1^2 - 1) + \cdots + [(n-1)^2 - 1] \\
&= n^2 - n + 2 - (n-1) + \frac{(n-1)n(2n-1)}{6} \\
&= \frac{2n^3 + 3n^2 - 11n + 18}{6}.
\end{aligned}$$

3. Borgov place des évêques blancs, et Beth Harmon place à son tour des évêques noirs sur un échiquier, en commencant par le blanc, de sorte qu'un nouvel évêque ne peut être placé sur une carré que s'il ne peut pas être " pris " par un évêque de l'autre couleur, déjà sur l'échiquier. Un joueur perd s'il ne peut pas placer d'évêque sur le plateau pendant son tour. Donnez une stratégie à Beth Harmon pour gagner.

Solution: Symétrie. Supposons que les carrés de l'échiquier soient étiquetés en coordonnées $\{1, ..., 8\} \times \{1, ..., 8\}$. Si Borgov joue un évêque à (m, n) alors Harmon répond à $(9-m, n)$. Vous devriez réfléchir aux raisons pour lesquelles cette stratégie fonctionne.

4. We begin with the set of integers $\{1, 2, \ldots, n\}$. We proceed by replacing any two integers $a \le b$ in the set with $b - a$, and then perform the same operation on this new set. Note that the new set may have the same integer repeated, but it will have one less element. Keep on doing this until there is just one integer left. Show how this integer will be odd or even depending on the value of $n \pmod 4$.

Solution: Let S_j be the jth set of integers, and suppose that they have sum s_j. If we get S_{j+1} by replacing $a \le b$ with $b-a$ then $s_{j+1} = s_j - a - b + (b-a) = s_j - 2a \equiv s_j \pmod 2$. Therefore the last element is $\equiv 1 + 2 + \ldots + n \pmod 2$ and this depends on $n \pmod 4$ since the sum of any four consecutive integers is even.

5. Given 11 red chips, 30 white chips and 19 blue chips, we can replace any two chips of two different colours, by two chips of the third colour. (For example, we may replace a white chip and a blue chip by two red chips.) Can we ever have the same number of chips of two different colours?

Solution: In our example we replace (r, w, b) by $(r-1, w+2, b-1)$. Therefore $b - w \bmod 3$ and $r - b \bmod 3$ are invariants, and are always $\equiv 1 \pmod 3$, and so r, b and w are always distinct mod 3, so no two can be equal.

6. We play the game of *number solitaire*: Start with a finite set S of distinct integers, with smallest element 0 and largest element n. If $m, m+1 \in S$ but $m+2 \notin S$ then we can remove m and $m+1$ from S and replace them by $m+2$. Show that we can keep on doing this until we obtain a set in which all the integers differ by at least 2, and the largest element is either n or $n+1$.

Solution: Let $\Sigma(S) := \sum_{m \in S} F_m$ where F_m is the mth Fibonacci number. When we replace $m, m+1$ by $m+2$, we do not change the value of $N(S)$. Suppose the sequence of sets is $S_1 = S, S_2, \ldots, S_k$ and let N_k be the largest element of S_k. By construction we have $n = N_1 \leq N_2 \leq \ldots \leq N_k$. We also have

$$F_{N_k} \leq \Sigma(S_k) = \Sigma(S) = \sum_{m \in S} F_m \leq \sum_{m \leq n} F_m = F_{n+2} - 1,$$

and so $n \leq N_k \leq n+1$. The values n and $n+1$ can be achieved, for example if $S = \{n\}$ and if $S = \{n-1, n\}$, respectively.

If $r, r+1, \ldots, r+\ell \in S$ but $r+\ell+1 \notin S$ then we can proceed by replacing $r+\ell-1, r+\ell$ by $r+\ell+1$, and then $r+\ell-3, r+\ell-2$ by $r+\ell-1$, etc. So eventually there will be no two elements that differ by 1 in our set.