Concours Putnam

Atelier de Pratique

Le mardi, 23 septembre 13h30-14h30

1. Find all integer solutions of the equation $a^2 + b^2 + c^2 = a^2b^2$.

Solution: We will prove that the only possible solution is with $a = b = c = 0$. Suppose that we have a solution (a, b, c) with at least one letter different from zero. Let 2^k be the highest power of 2 that divides a, b, c and write $a = 2^k a_1$, $b = 2^k b_1$, $c = 2^k c_1$. Now we get $a_1^2 + b_1^2 + c_1^2 = 2^{2k} a_1^2 b_1^2$. At least one of a_1, b_1, c_1 is odd. Since squares are congruent to 0 or 1 modulo 4, it follows that the left-hand side is congruent to 1,2,3 modulo 4. The right-hand side in congruent to 0 modulo 4 unless $k = 0$ and a_1, b_1 are odd. In that case, the right-hand side is congruent to 1 modulo 4, but the left-hand side is congruent to 2 or 3 modulo 4 (depending on whether c_1 is odd or even) so we get a contraditcion.

2. In a round-robin tournament with n players, P_1, P_2,... ,P_n, each of the players plays a match against every other player. There are no ties, so each match ends in a win for one side and a loss for the other side. Let W_k denote the number of wins of player P_k , and let L_k denote the number of losses of P_k. Show that $\sum_{k=1}^n W_k^2 = \sum_{k=1}^n L_k^2$.

Solution: The identity to be shown can be written as $\sum_{k=1}^n (W_k - L_k)(W_k + L_k) = 0$. Now, for each k, $W_k + L_k$ is equal to the total number of games played by P_k , i.e., $n-1$. Factoring out $n-1$, the identity that we have to prove becomes $\sum_{k=1} (W_k - L_k) = 0$, or equivalently $\sum_{k=1}^n W_k = \sum_{k=1}^n L_k$. But this is obvious, since the total number of wins must equal the total number of losses in the tournament.

3. Let $P(x)$ be a polynomial of degree n satisfying $P(k) = k$ for $k = 1, \ldots, n$ and $P(0) = 1$. Find $P(-1)$.

Solution: Let $Q(x) = P(x) - x$. Then $Q(x)$ is a polynomial of degree n whose roots are $k = 1, 2, \ldots, n$. Thus $Q(x)$ is of the form $Q(x) = c\prod_{k=1}^n (x - k)$ for some constant c, and $P(x) = x + c\prod_{k=1}^n (x - k)$. Setting $x = 0$ gives $1 = P(0) = c(-1)^n n!$, so $c = \frac{(-1)^n}{n!}$. Hence,

$$P(-1) = -1 + \frac{(-1)^n}{n!} \prod_{k=1}^n (-1 - k) = -1 + (n + 1) = n.$$

4. Find the smallest positive integer n such that every digit of $15n$ is either 8 or 0.

Solution: Let $N = 15n$. Since the number N is divisible by 3 and 5, the sum of its digits must be divisible by 3, and the last digit must be 0 or 5. If N consists only of digits 0 and 8, it follows that the last digit must be 0 and the number of digits 8 contained in N must be multiple of 3. The smallest positive number with these properties is $N = 8880$, so $n = N/15 = 592$ is the smallest positive integer such that every digit of $15n$ is 8 or 0.

5. On a table there are 100 tokens. Taking turns two players remove 5, 6, 7, 8, 9 or 10 tokens, at their choice. The player that removes the last token wins. Find a winning strategy and determine which player will be the winner.

Solution: The first player takes 10 tokens leaving 90 tokens on the table. Then if the second player takes n tokens $5 \leq n \leq 10$, the first player takes $15 - n$, so the will remove 15 tokens together. Since 90 is a multiple of 15, the first player will take the last token and win.

6. Let f be a function from the positive integers into the positive integers and satisfying $f(n+1) > f(n)$ and $f(f(n)) = 3n$ for all n. Find $f(100)$.

Solution: We will show that, for any integers $k \geq 0$ and $0 \leq m < 3^k$, $f(3^k + m) = 2 \cdot 3^k + m$. Notice that the first condition implies $f(n+m) \geq f(n) + m$ for any positive integers n and m. The second condition implies $f(f(1)) = 3$. If $f(1) > 2$ then we have $3 = f(f(1)) \geq f(f(1) - 1) + 1 \geq f(2) + 1 \geq f(1) + 2$, which is a contradiction. If $f(1) = 1$, we have $3 = f(f(1)) = f(1) = 1$ contradiction. Then we must have $f(1) = 2$.This implies that $f(2) = f(f(1)) = 3$. We prove by induction that for k nonnegative integer, $f(2 \cdot 3^k) = 3^{k+1}$ and $f(3^k) = 2 \cdot 3^k$. For $k = 0$ this has been already proved. Assume that both statements hold for k. Then $f(3^{k+1}) = f(f(2 \cdot 3^k)) = 3 \cdot 2 \cdot 3^k = 2 \cdot 3^{k+1}$ and $f(2 \cdot 3^{k+1}) = f(f(3^{k+1})) = 3 \cdot 3^{k+1} = 3^{k+2}$ which completes the induction. We see that the values $f(3^k + m)$ for $m = 0, 1, \ldots, 3^k - 1$ must form an increasing sequence of 3^k distinct integers, all contained in the interval $[2 \cdot 3^k, 3 \cdot 3^k - 1]$. Since there are exactly 3^k integers in that interval, these values must fill the entire interval and we have $f(3^k + m) = 2 \cdot 3^k + m$.

Therefore, $f(100) = f(3^4 + 19) = 2 \cdot 3^4 + 19 = 162 + 19 = 181$.