Concours Putnam

Atelier de Pratique

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Fonctions

1. The functions $f(x) = 4x - 4x^2$ and $\sin \pi x$ agree at $x = 0, 1/2$, and 1. Show that $f(x) \geq \sin \pi x$ for $0 \leq x \leq 1$.

> **Solution:** Since $f(x)$ and $\sin \pi x$ are symmetric about $x = 1/2$, it suffices to prove the inequality for $0 \leq x \leq 1/2$. Let $g(x) = f(x) - \sin \pi x$. We have that $g'(x) = 4 - 8x - \pi \cos \pi x$ and $g''(x) = -8 + \pi^2 \sin \pi x$. Thus $g''(x)$ increases monotonically from -8 to $\pi^2 - 8 > 0$ as x ranges over the interval $[0, 1/2]$ and therefore has a unique zero x_0 in this interval, an inflection point for $g(x)$. It follows that $g'(x)$ is decreasing for $0 \leq x \leq x_0$ and increasing for $x_0 < x \leq 1/2$. Thus g' has a unique minimum at x_0.
>
> Note that $g(0) = g(1/2) = 0$, and $g'(0) = 4 - \pi > 0$ so $g(x) > 0$ for $x > 0$ close enough to 0.
>
> By Rolle's theorem, there can be at most one additional zero $g(z) = 0$ with $0 < z < 1/2$, or else g'' would have at least two zeros in the interval. If there are no zeros, we are done. Assume thus in contradiction there is another zero z of g in the interval $[0, 1/2]$, so $g(x) > 0$ for $0 < x < z$, and $g(x) < 0$ for $z < x < 1/2$. Then for $max(z, x_0) < x < 1/2$, $g(x) < 0$ and g us convex. Since $g(1/2) = 0$, g must be increasing at some point $x_2 \in (max(z, x_0), 1/2)$, and since g' is increasing, $g' > 0$ also in $[x_2, 1/2]$. But then $g'(1/2) > 0$, a contradiction.

2. Determine, with proof, all functions f defined on the set of integers and satisfying

$$f(n + m) + f(n - m) = 2(f(m) + f(n))$$

for all n and m.

> **Solution:** Setting $m = n = 0$ gives $2f(0) = 4f(0)$ which implies $f(0) = 0$. Setting $n = 0$, $f(m) + f(-m) = 2(f(m) + f(0)) = 2f(m)$, which implies that $f(-m) = f(m)$ for all m. Let $\alpha = f(1)$ and apply the equation for $m = 1$. $f(n + 1) + f(n - 1) = 2(\alpha + f(n))$, or $f(n+1) = 2f(n) - f(n-1) + 2\alpha$ for all n. By induction one can prove that $f(n) = \alpha n^2$ for all positive n which also holds for the negatives. Conversely, any function of the form $f(n) = \alpha n^2$ satisfies the equation.

3. Supposed that $a_0, \ldots, a_n \in \mathbb{R}$ and $0 < x < 1$ satisfy

$$\frac{a_0}{1-x} + \frac{a_1}{1-x^2} + \ldots + \frac{a_n}{1-x^{n+1}} = 0.$$

Prove there is $0 < y < 1$ s.t. $P(y) := a_0 + a_1 y + \ldots + a_n y^n = 0$.

Solution: Assume the contrary. By the intermediate value theorem, $P(y)$ maintains its sign in $[0,1]$, and we may change the sign of all a_j to assume $P(y) > 0$. Thus for all $\alpha > 0$ we have $P(x^\alpha) > 0$, and also $P(1) \geq 0$ by continuity. Thus also $\sum_{j=0}^n a_j x^{kj+k} \geq 0$ over all $k \geq 0$, and summing over all $k \geq 0$ we find

$$\sum_{j=0}^{n} \frac{a_j}{1 - x^{j+1}} > 0,$$

in contradiction.

4. Find all differentiable $f : \mathbb{R} \to \mathbb{R}$ such that for all real x and natural n,

$$f'(x) = \frac{f(x+n) - f(x)}{n}$$

Solution: One finds that $f'(x+1) = f'(x)$. Hence $f'(x) = f(x+1) - f(x)$ is constant.

5. Evaluate $\int_0^{\frac{\pi}{2}} \ln(\sin x)dx$.

Solution: Let $I = \int_0^{\frac{\pi}{2}} \ln(\sin x)dx$. By making $y = \frac{\pi}{2} - x$, and using $\sin\left(\frac{\pi}{2} - y\right) = \cos y$, we see that $I = \int_0^{\frac{\pi}{2}} \ln(\cos y)dy$. Thus,

$$2I = \int_0^{\frac{\pi}{2}} (\ln \sin x + \ln \cos x)dx = \int_0^{\frac{\pi}{2}} \ln(\sin x \cos x)dx$$

$$= \int_0^{\frac{\pi}{2}} \ln\left(\frac{1}{2}\sin(2x)\right)dx = \left(\ln\frac{1}{2}\right)\frac{\pi}{2} + \int_0^{\frac{\pi}{2}} \ln \sin(2x)dx.$$

The change of variables $y = 2x$ shows that the last integral equals I. Solving the resulting equation yields $I = -\frac{\pi}{2}\ln 2$.

6. Let

$$I_\alpha = \int_0^\infty \frac{dx}{x^\alpha(1+x)}, \quad 0 < \alpha < 1.$$

Find the choice of α that minimizes I_α. Explain.

Solution: We will show that the minimum occurs when $\alpha = \frac{1}{2}$. Split I_α into $\int_0^1$ and $\int_1^\infty$. Setting $u = \frac{1}{x}$, $du = -\frac{dx}{x^2}$ in the first integral leads to

$$\int_0^1 = \int_1^\infty \frac{du}{u^2 u^{-\alpha}\left(1+\frac{1}{u}\right)} = \int_1^\infty \frac{du}{u^{1-\alpha}(u+1)}.$$

Hence,

$$I_\alpha = \int_0^1 + \int_1^\infty = \int_1^\infty (x^{-\alpha} + x^{\alpha-1})\frac{dx}{x+1}.$$

To show that I_α is minimal at $\alpha = 1/2$ one could take the arithmetic-geometric mean inequality which gives

$$\frac{x^{-\alpha} + x^{\alpha-1}}{2} \geq \frac{1}{\sqrt{x}}$$

The equality is attained when $x^{-\alpha} = x^{\alpha-1}$ which corresponds to $\alpha = 1/2$.

7. Let f be a continuous, decreasing function on $[0,1]$. Show that

$$\int_0^1 f(x)(1-2x)dx \geq 0.$$

Solution: Splitting the range of integration in two parts $0 \leq x \leq 1/2$ and $1/2 \leq x \leq 1$ and making the change of variables $y = 1-x$ in the integral over the latter range, the given integral can be written as

$$\int_0^{1/2} f(x)(1-2x)dx + \int_0^{1/2} f(1-y)(2y-1)dy = \int_0^{1/2} (f(x) - f(1-x))(1-2x)dx.$$

Since f is decreasing, we have $f(x) - f(1-x) \geq 0$ for $0 \leq x \leq 1/2$. Hence the integrand in the last integral is nonnegative in the range of integration, and the integral is therefore nonnegative as well.

8. Evaluate

$$\int_0^\infty \frac{\arctan(\pi x) - \arctan(x)}{x} dx.$$

Solution: This is (Putnam 1982, A3). Answer $\frac{\pi}{2}\ln\pi$.

$$\int_0^\infty \frac{\arctan(\pi x) - \arctan(x)}{x} dx$$

$$\int_0^\infty \int_1^\pi \frac{dy}{1+(xy)^2}dx = \int_1^\pi \int_0^\infty \frac{dx}{1+(xy)^2}dy$$
$$\int_1^\pi \left.\frac{\arctan(xy)}{y}\right|_{x=0}^{x=\infty} dy = \frac{\pi}{2}\int_1^\pi \frac{dy}{y} = \frac{\pi}{2}\ln\pi$$

9. Let T be the triangle with vertices $(0,0), (a,0)$, and $(0,a)$. Find

$$\lim_{a\to\infty} a^4 e^{-a^3}\int_T e^{x^3+y^3}\,dxdy.$$

Solution: This is (Putnam 1983, A6). Answer: $\frac{2}{9}$.

The corresponding indefinite integral is intractable and the definite integral diverges. But $\frac{e^{a^3}}{a^4}$ also diverges. So we have the ratio of two divergent quantities. We apply l'Hôpital's rule.

The first step is to rearrange the integral so that a only occurs as an integration limit for one of the variables (thus making it easier to differentiate with respect to it).

After a little experimentation we take $s = x+y$, $t = x-y$. The Jacobian is $\frac{1}{2}$ and so we get

$$\frac{1}{2}\int_0^a \int_{-s}^s e^{\frac{s^3}{4}+\frac{3st^2}{4}}\,dtds.$$

Differentiating with respect to a, we get

$$\frac{1}{2}\int_{-a}^a e^{\frac{a^3}{4}+\frac{3at^2}{4}}\,dt = \frac{1}{2}e^{\frac{a^3}{4}}\int_{-a}^a e^{\frac{3at^2}{4}}\,dt.$$

Similarly, differentiating the denominator gives $\left(\frac{3}{a^2}-\frac{4}{a^5}\right)e^{a^3}$. We can cancel out the $e^{\frac{a^3}{4}}$ to get

$$\frac{\int_{-a}^a e^{\frac{3at^2}{4}}\,dt}{2\left(\frac{3}{a^2}-\frac{4}{a^5}\right)e^{\frac{3a^3}{4}}}$$

but both these still diverge. Accordingly, we must apply the rule again.

We would like to eliminate the a in the integrand to make differentiation simpler. This can be achieved by setting $s = a^{1/2}t$. Notice that the integrand has the same value for t and $-t$ (or s and $-s$) so we can further simplify by taking the integration from 0 to a and doubling. Thus we get

$$\frac{a^{-1/2}\int_0^{a^{3/2}} e^{\frac{3s^2}{4}}\,ds}{\left(\frac{3}{a^2}-\frac{4}{a^5}\right)e^{\frac{3a^3}{4}}} = \frac{\int_0^{a^{3/2}} e^{\frac{3s^2}{4}}\,ds}{\left(\frac{3}{a^{3/2}}-\frac{4}{a^{9/2}}\right)e^{\frac{3a^3}{4}}}.$$

Now differentiating the numerator and the denominator give

$$\frac{\frac{3a^{1/2}}{2}e^{\frac{3a^3}{4}}}{\left(-\frac{9}{2a^{5/2}}+\frac{18}{a^{11/2}}\right)e^{\frac{3a^3}{4}}+\left(\frac{3}{a^{3/2}}-\frac{4}{a^{9/2}}\right)e^{\frac{3a^3}{4}}\frac{9a^2}{4}}=\frac{\frac{3a^{1/2}}{2}}{\left(-\frac{9}{2a^{5/2}}+\frac{18}{a^{11/2}}\right)+\left(\frac{27a^{1/2}}{4}-\frac{9}{a^{5/2}}\right)}.$$

This evaluates to $\frac{2}{9}$ as a tends to infinity.