Concours Putnam

Atelier de Pratique
Le jeudi, 20 novembre 13h30-14h30
5448 Pav. André Aisenstadt

Fonctions

1. The functions $f(x) = 4x - 4x^2$ and $\sin \pi x$ agree at $x = 0, 1/2$, and 1. Show that $f(x) \geq \sin \pi x$ for $0 \leq x \leq 1$.

2. Determine, with proof, all functions f defined on the set of integers and satisfying

$$f(n+m) + f(n-m) = 2(f(m) + f(n))$$

for all n and m.

3. Supposed that $a_0, \ldots, a_n \in \mathbb{R}$ and $0 < x < 1$ satisfy

$$\frac{a_0}{1-x} + \frac{a_1}{1-x^2} + \ldots + \frac{a_n}{1-x^{n+1}} = 0.$$

Prove there is $0 < y < 1$ s.t. $P(y) := a_0 + a_1 y + \ldots + a_n y^n = 0$.

4. Find all differentiable $f : \mathbb{R} \to \mathbb{R}$ such that for all real x and natural n,

$$f'(x) = \frac{f(x+n) - f(x)}{n}$$

5. Evaluate $\int_0^{\frac{\pi}{2}} \ln(\sin x) dx$.

6. Let

$$I_\alpha = \int_0^\infty \frac{dx}{x^\alpha (1+x)}, \quad 0 < \alpha < 1.$$

Find the choice of α that minimizes I_α. Explain.

7. Let f be a continuous, decreasing function on $[0, 1]$. Show that

$$\int_0^1 f(x)(1-2x) dx \geq 0.$$

8. Evaluate

$$\int_0^\infty \frac{\arctan(\pi x) - \arctan(x)}{x} dx.$$

9. Let T be the triangle with vertices $(0, 0), (a, 0)$, and $(0, a)$. Find

$$\lim_{a \to \infty} a^4 e^{-a^3} \int_T e^{x^3 + y^3} dx dy.$$