Concours Putnam

Atelier de Pratique

Le jeudi, 5 novembre 13:30h-14:30h (Salle: Pavillon André-Aisenstadt 5448)

Inequalités

1. Arithmetic Mean, Geometric Mean, Harmonic Mean Inequalities. Let $a_1, \ldots, a_n$ be positive numbers. Then the following inequalities hold:

$$\frac{n}{\frac{1}{a_1} + \ldots + \frac{1}{a_n}} \leq \sqrt[n]{a_1 a_2 \ldots a_n} \leq \frac{a_1 + \ldots + a_n}{n}$$

(Harmonic Mean $<$ Geometric Mean $<$ Arithmetic Mean).

In all cases equality holds if and only if $a_1 = \ldots = a_n$.

2. Power Means Inequality. Let r be a non-zero real number. We define the r-mean or rth power mean of positive numbers $a_1, \ldots, a_n$ as follows:

$$M_r(a_1, \ldots, a_n) = \left(\frac{1}{n} \sum_{i=1}^{n} a_i^r \right)^{1/r}$$

M_1 =arithmetic mean, M_2 = quadratic mean, M_{-1} = harmonic mean.

Furthermore we define the 0-mean to be equal to the geometric mean:

$$M_0(a_1, \ldots, a_n) = \left(\prod_{i=1}^{n} a_i \right)^{1/n}.$$

Then, for any real numbers r, s such that $r < s$, the following inequality holds:

$$M_r(a_1, \ldots, a_n) \leq M_s(a_1, \ldots, a_n).$$

Equality holds if and only if $a_1 = \ldots = a_n$.

2.1. Power Means Sub/Superadditivity. Let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be positive real numbers.

(a) If $r > 1$, then the r-mean is subadditive, i.e.:

$$M_r(a_1 + b_1, \ldots, a_n + b_n) \leq M_r(a_1, \ldots, a_n) + M_r(b_1, \ldots, b_n).$$

(b) If $r < 1$, then the r-mean is superadditive, i.e.:

$$M_r(a_1 + b_1, \ldots, a_n + b_n) \geq M_r(a_1, \ldots, a_n) + M_r(b_1, \ldots, b_n).$$

Equality holds if and only if $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ are proportional.

3. Cauchy-Schwarz.

$$\left(\sum_{i=1}^{n} a_i b_i \right)^2 \leq \left(\sum_{i=1}^{n} a_i^2 \right) \left(\sum_{i=1}^{n} b_i^2 \right).$$

4. Hölder. If $p > 1$ and $1/p + 1/q = 1$ then

$$\left|\sum_{i=1}^{n} a_i b_i\right| \le \left(\sum_{i=1}^{n} |a_i|^p\right)^{1/p} \left(\sum_{i=1}^{n} |b_i|^q\right)^{1/q}.$$

(For $p = q = 2$ we get Cauchy-Schwarz.)

5. Minkowski. If $p > 1$ then

$$\left(\sum_{i=1}^{n} |a_i + b_i|^p\right)^{1/p} \le \left(\sum_{i=1}^{n} |a_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |b_i|^p\right)^{1/p}.$$

Equality holds iff $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ are proportional.

6. Norm Monotonicity. If $a_i > 0$ $(i = 1, \ldots, n)$, $s > t > 0$, then

$$\left(\sum_{i=1}^{n} |a_i|^s\right)^{1/s} \le \left(\sum_{i=1}^{n} |a_i|^t\right)^{1/t}.$$

7. Chebyshev. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be sequences of real numbers which are monotonic in the same direction (we have $a_1 \le a_2 \le \ldots \le a_n$ and $b_1 \le b_2 \le \ldots \le b_n$, or we could reverse all inequalities.) Then

$$\frac{1}{n}\sum_{i=1}^{n} a_i b_i \ge \left(\frac{1}{n}\sum_{i=1}^{n} a_i\right)\left(\frac{1}{n}\sum_{i=1}^{n} b_i\right).$$

8. Schur. If x, y, z are positive real numbers and k is a real number such that $k \ge 1$, then

$$x^k(x-y)(x-z) + y^k(y-x)(y-z) + z^k(z-x)(z-y) \ge 0.$$

For $k = 1$ the inequality becomes

$$x^3 + y^3 + z^3 + 3xyz \ge xy(x+y) + yz(y+z) + zx(z+x).$$

9. Weighted Power Means Inequality. Let $w_1, \ldots, w_n$ be positive real numbers such that $w_1 + \ldots + w_n = 1$. Let r be a non-zero real number. We define the rth weighted power mean of non-negative numbers $a_1, \ldots, a_n$ as follows:

$$M_{r,w}(a_1, \ldots, a_n) = \left(\sum_{i=1}^{n} w_i a_i^r\right)^{1/r}.$$

As $r \to 0$ the rth weighted power mean tends to:

$$M_{0,w}(a_1, \ldots, a_n) = \left(\prod_{i=1}^{n} a_i\right).$$

which we call 0th weighted power mean. If $w_i = 1/n$ we get the ordinary rth power means. Then for any real numbers r, s such that $r < s$, the following inequality holds:

$$M_{r,w}(a_1, \ldots, a_n) \le M_{s,w}(a_1, \ldots, a_n).$$

Questions

1. If $a, b, c > 0$, prove that $(a^2b + b^2c + c^2a)(ab^2 + bc^2 + ca^2) \geq 9a^2b^2c^2$.

2. If $a, b, c \geq 0$, prove that $\sqrt{3(a+b+c)} \geq \sqrt{a} + \sqrt{b} + \sqrt{c}$.

3. Show that

$$\sqrt{a_1^2 + b_1^2} + \sqrt{a_2^2 + b_2^2} + \ldots + \sqrt{a_n^2 + b_n^2} \geq \sqrt{(a_1 + a_2 + \ldots + a_n)^2 + (b_1 + b_2 + \ldots + b_n)^2}$$

4. Find the minimum value of the function $f(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \ldots + x_n$, where $x_1, x_2, \ldots, x_n$ are positive real numbers such that $x_1 x_2 \ldots x_n = 1$.

5. Prove that in a triangle with sides a, b, c and opposite angles A, B, C (in radians) the following relation holds:
$$\frac{aA + bB + cC}{a + b + c} \geq \frac{\pi}{3}.$$

6. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ nonnegative real numbers. Show that

$$(a_1 a_2 \ldots a_n)^{1/n} + (b_1 b_2 \ldots b_n)^{1/n} \leq ((a_1 + b_1)(a_2 + b_2) \ldots (a_n + b_n))^{1/n}$$

7. For $a_1, \ldots, a_n > 0$ prove

$$(1 + \frac{a_1^2}{a_2})(1 + \frac{a_2^2}{a_3}) \cdots (1 + \frac{a_n^2}{a_1}) \geq (1 + a_1)(1 + a_2) \cdots (1 + a_n)$$

8. Which is larger, 2025^{2025} or 2026^{2024}?

9. Find the minimum value of

$$(u - v)^2 + \left(\sqrt{2 - u^2} - \frac{9}{v}\right)^2$$

for $0 < u < \sqrt{2}$ and $v > 0$.

10. Let m and n be positive integers. Show that

$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!n!}{m^m n^n}.$$