Concours Putnam

Atelier de Pratique

Le jeudi, 26 septembre 12h30-13h30

Binômes

Binomial theorem

$$\sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n \qquad n = 1, 2 \ldots.$$

Binomial series

$$\sum_{k=0}^{\infty} \binom{\alpha}{k} x^k = (1+x)^\alpha \quad |x| < 1, \alpha \text{ any real number}$$

Pascal's identity

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k} \text{ for } k, n \text{ positive integers}$$

1. Compute the following

    1. $\sum_{k=0}^{n} \binom{n}{k}$
    2. $\sum_{k=0}^{n} (-1)^k \binom{n}{k}$
    3. $\sum_{k=0}^{2n} (-1)^k k^n \binom{2n}{k}$
    4. $\sum_{k=0} \binom{n}{k}^2$
    5. $\sum_{k=0}^{n} \frac{1}{k+1} \binom{n}{k}$
    6. $\sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k}$
    7. $\sum_{m=0}^{n} \binom{m}{k}$
    8. $\sum_{k=0}^{n} \frac{\binom{m}{k}}{\binom{n}{k}} \quad n \geq m$
    9. $\sum_{k=0}^{n} \binom{2k}{k} \binom{2n-2k}{n-k}$

Solution:

1. 2^n. Interpret the sum as the number of subsets of an n-element set or apply the binomial theorem with $x = 1$.
2. 1 if $n = 0$ and 0 if $n \geq 1$. Apply the binomial theorem with $x = -1$.

3. 0 if $n \geq 1$.

   Write
   $$f(x) = (1 - e^x)^{2n} = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} e^{kx}$$
   differentiate n times and evaluate in $x = 0$. (In fact, it works for any $r < 2n$)

4. $\binom{2n}{n}$. Write the term as $\binom{n}{k}\binom{n}{n-k}$ and apply item 6 (Vandermonde's identity).

5. $\frac{2^{n+1}-1}{n+1}$. Use that $\frac{1}{k+1}\binom{n}{k} = \frac{1}{n+1}\binom{n+1}{k+1}$. Or start from the Binomial theorem and integrate with x between 0 and 1.

6. $\binom{m+n}{r}$. This is the Vandermonde identity. Count the number of ways to form a group of r people from a set of m men and n women.

7. $\binom{n+1}{k+1}$. Do induction, using Pascal's identity. Or consider all the $k+1$ subsets of the set $\{1, 2, \ldots, n+1\}$ and group them according to their greatest element.

8. $\frac{n+1}{n-m+1}$. First prove the identity $\frac{\binom{m}{k}}{\binom{n}{k}} = \frac{\binom{n-k}{n-m}}{\binom{n}{m}}$ and use the previous problem.

9. For any real α one defines
   $$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!},$$
   extending the binomial coefficient (the Euler Beta function is a further generalization to paris of reals, but it takes integrals to define).

   Check that
   $$\binom{-1/2}{n} = (-1)^n 4^n \binom{2n}{n}.$$
   The sum in the question is now just a multiple of the Vandermonde sum (6)
   $$4^n(-1)^n \sum_{k0}^{n} \binom{-1/2}{k}\binom{-1}{n-k} = 4^n$$
   (but we should understand why (6) is valid for real m, n)

2. How many subsets that have an even number of elements are there in a set with n elements?

Solution: 2^{n-1} if $n \geq 1$. If S a subset with an even number of elements, the membership of the first $n-1$ elements in S can be decided arbitrarily, but the membership of the last element in S is completely determined by the previous $n-1$ choices. Or by problem 1.2, the number of subsets with an even number of elements is equal to the number of subsets with an odd number of elements.

3. In how many ways can 16 players be paired for the first round of a tennis tournament?

Solution:

$$\frac{\binom{16}{2}\binom{14}{2}\cdots\binom{2}{2}}{8!} = 15 \cdot 13 \cdot 11 \cdots 3 \cdot 1$$

The first player can be paired with any of the 15 remaining players. Now we are left with a smaller problem of pairing 14 players, so we repeat the process or appeal to induction.

Another interpretation consists of taking into account the relative positions of each couple and all the games. We draw the tree and divide by all the symmetries, in order to get

$$\frac{16!}{2^8 \cdots 2^4 \cdot 2^2 \cdot 2^1}.$$

Warning: this number is different from the other, and possibly gives the best answer.

4. How many ways are there to place an order of n donuts if there are k varieties to choose from?

Solution: Imagine the donuts lined up with $k-1$ dividers between k the different varieties, for a total of $n+k-1$ spots: $k-1$ for the dividers, and n for the donuts. Then count the number of ways to pick the $k-1$ spots for the dividers out of the $n+k-1$ available spots.

5. How many 10 letter "words" can be formed using 3 A's, 2 E's, 2 I's, one B, one C, and one D?

Solution:

$$\binom{10}{3}\binom{7}{2}\binom{5}{2}\binom{3}{1}\binom{2}{1}\binom{1}{1}$$

First pick three out of the 10 available slots for the letters and place the A's in those slots, then pick two out of the remaining 7 free slots for the E's, and so on.

6. How many ordered triples of sets (A, B, C) satisfy $A \cap B \cap C = \emptyset$ and $A \cup B \cup C = \{1, 2, ..., 10\}$?

Solution: This is (Putnam '85, A1). Consider the Venn diagram formed by A, B and C. Each element of $\{1, 2, ..., 10\}$ can, independently of the others, go into six of the eight regions in the diagram. Thus there are 6^{10} triples (A, B, C) with the specified properties.

7. Let $1 \leq r \leq n$ and consider all subsets of r elements of the set $\{1, 2, \dots, n\}$. Each of these subsets has a smallest member. Let $F(n, r)$ denote the arithmetic mean of these smallest numbers; prove that
$F(n, r) = \frac{n+1}{r+1}$.

Solution: (IMO 1981) Clearly

$$F(n,r) = \frac{\binom{n-1}{r-1} + 2\binom{n-2}{r-1} + \ldots + (n-r+1)\binom{r-1}{r-1}}{\binom{n}{r}}.$$

The numerator can be computed by

$$\sum_{j=1}^{n} j\binom{n-j}{r-1} = \sum_{j=1}^{n-r+1} j\binom{n-j}{r-1} = \sum_{i=1}^{n-r+1}\sum_{j=i}^{n-r+1}\binom{n-j}{r-1}$$

$$= \sum_{i=1}^{n-r+1}\binom{n-i+1}{r} = \binom{n+1}{r+1}$$

from which $F(n, r)$ quickly follows.

8. Show that the coefficient of x^k in the expansion of $(1 + x + x^2 + x^3)^n$ is $\sum_{j=0}^{k}\binom{n}{j}\binom{n}{k-2j}$.

Solution: This is (Putnam '92, B2). Write $1 + x + x^2 + x^3 = (1 + x)(1 + x^2)$. So the expression is

$$\left(\sum\binom{n}{i}x^i\right)\left(\sum\binom{n}{j}x^{2j}\right).$$

To get a term in x^k we must multiply a term x^{2j} by a term x^{k-2j} for some j. The result follows.

9. Let p be a prime > 2. Prove that $\sum_{0\le n\le p}\binom{p}{n}\binom{p+n}{n} \equiv 2^p + 1 \pmod{p^2}$.

Solution: This is (Putnam '91, B4).

Let $D_n = \binom{p}{n}\binom{p+n}{n}$. We show that $D_0 \equiv \binom{p}{0}^2 \pmod{p^2}$, $D_p \equiv \binom{p}{p} + 1 \pmod{p^2}$, and for the other terms $D_n \equiv \binom{p}{n} \pmod{p^2}$. The result then follows since $\sum_{n=0}^{p}\binom{p}{n} = 2^p$.

For $n = 0$, it is obvious that $D_0 = 1 = \binom{p}{0}$.

For $n = p$. Notice that we can write $(p-1)!\binom{2p-1}{p-1}$ as $(1+p)(2+p)(3+p)\cdots(p-1+p)$. Expanding, we get $(p-1)! + (1+2+\cdots+p-1)p + O(p^2) \equiv (p-1)! \pmod{p^2}$, since $(1+2+\cdots+p-1) = \frac{p(p-1)}{2}$. Hence, $\binom{2p-1}{p-1} \equiv 1 \pmod{p^2}$. Hence $\binom{2p}{p} = \frac{2p}{p}\binom{2p-1}{p-1} \equiv 2 \equiv 1 + \binom{p}{p} \pmod{p^2}$.

For $0 < n < p$, $\binom{p}{n} \equiv 0 \pmod{p}$ [because all the factors in $n!(p-n)!$ are $< p$ and hence relatively prime to p, so the factor p in the numerator $p!$ remains after dividing by $n!(p-n)!$]. Also $p+i \equiv i \not\equiv 0 \pmod{p}$, so $(p+n)\cdots(p+2)(p+1) \equiv n! \pmod{p}$ and hence $\binom{p+n}{n} \equiv 1 \pmod{p}$ for $0 < n < p$. In other words, $\binom{p+n}{n} = kp+1$ for some integer k. But $kp\binom{p}{n} \equiv 0 \pmod{p^2}$, since p divides $\binom{p}{n}$, so $D_n = \binom{p}{n}\binom{p+n}{n} \equiv \binom{p}{n} \pmod{p^2}$.

10. Let p be a prime ≥ 5. Prove that p^2 divides $\sum_{r=1}^{[2p/3]}\binom{p}{r}$.

Solution: This is (Putnam '96, A5).

Each $\binom{p}{i}$ is divisible by p. Write $k = [2p/3]$. So we have to show that $S = \frac{\binom{p}{1}}{p} + \frac{\binom{p}{2}}{p} + \ldots + \frac{\binom{p}{k}}{p}$ is divisible by p. We work in the field mod p. $\frac{\binom{p}{i}}{p} = \frac{(p-1)\cdots(p-i+1)}{1\cdot 2\cdots i} \equiv \frac{(-1)^{i-1}}{i} \pmod{p}$. Write $h = \lfloor k/2 \rfloor = \lfloor p/3 \rfloor$. Hence $S \equiv 1 - \frac{1}{2} + \cdots \pm \frac{1}{k} \equiv 1 + \frac{1}{2} + \cdots + \frac{1}{k} - 2\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2h}\right) \equiv 1 + \frac{1}{2} + \cdots + \frac{1}{k} - \left(1 + \frac{1}{2} + \cdots + \frac{1}{h}\right) \equiv 1 + \frac{1}{2} + \cdots + \frac{1}{k} + \left(\frac{1}{p-1} + \frac{1}{p-2} + \cdots + \frac{1}{p-h}\right) \pmod{p}$. By considering $p \equiv 1, 2 \pmod{3}$ separately, we can easily check that $p - h \equiv k + 1$ and hence $S \equiv 1 + \frac{1}{2} + \cdots + \frac{1}{p-1} \pmod{p}$, which is a complete set of reduced residues and hence sums to zero.