Concours Putnam

Atelier de Pratique

Le jeudi, 26 septembre 12h30-13h30

Binômes

Binomial theorem

$$\sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n \qquad n = 1, 2 \ldots.$$

Binomial series

$$\sum_{k=0}^{\infty} \binom{\alpha}{k} x^k = (1+x)^\alpha \quad |x| < 1, \alpha \text{ any real number}$$

Pascal's identity

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k} \text{ for } k, n \text{ positive integers}$$

1. Compute the following

   1. $\sum_{k=0}^{n} \binom{n}{k}$
   2. $\sum_{k=0}^{n} (-1)^k \binom{n}{k}$
   3. $\sum_{k=0}^{2n} (-1)^k k^n \binom{2n}{k}$
   4. $\sum_{k=0} \binom{n}{k}^2$
   5. $\sum_{k=0}^{n} \frac{1}{k+1} \binom{n}{k}$
   6. $\sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k}$
   7. $\sum_{m=0}^{n} \binom{m}{k}$
   8. $\sum_{k=0}^{n} \frac{\binom{m}{k}}{\binom{n}{k}} \quad n \geq m$
   9. $\sum_{k=0}^{n} \binom{2k}{k} \binom{2n-2k}{n-k}$

2. How many subsets that have an even number of elements are there in a set with n elements?

3. In how many ways can 16 players be paired for the first round of a tennis tournament?

4. How many ways are there to place an order of n donuts if there are k varieties to choose from?

5. How many 10 letter "words" can be formed using 3 A's, 2 E's, 2 I's, one B, one C, and one D?

6. How many ordered triples of sets (A, B, C) satisfy $A \cap B \cap C = \emptyset$ and $A \cup B \cup C = \{1, 2, ..., 10\}$?

7. Let $1 \leq r \leq n$ and consider all subsets of r elements of the set $\{1, 2, \ldots, n\}$. Each of these subsets has a smallest member. Let $F(n, r)$ denote the arithmetic mean of these smallest numbers; prove that

   $F(n, r) = \frac{n+1}{r+1}$.

8. Show that the coefficient of x^k in the expansion of $(1 + x + x^2 + x^3)^n$ is $\sum_{j=0}^{k} \binom{n}{j}\binom{n}{k-2j}$.

9. Let p be a prime > 2. Prove that $\sum_{0 \leq n \leq p} \binom{p}{n}\binom{p+n}{n} \equiv 2^p + 1 \pmod{p^2}$.

10. Let p be a prime ≥ 5. Prove that p^2 divides $\sum_{r=1}^{[2p/3]} \binom{p}{r}$.