# Concours Putnam

Atelier de Pratique
Le jeudi, 7 novembre 12h30-13h30
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## Théorie des nombres

**Euler Totient Function** The Euler totient function $\phi(n)$, denoting the number of positive integers not exceeding $n$ and relatively prime to $n$ is given by

$$\phi(n) = n \prod_{p_i | n} \left(1 - \frac{1}{p_i}\right)$$

where the $p_i$'s are prime numbers.

**Euler's Theorem** If $a$ and $n$ are relatively prime integers, then $a^{\phi(n)} \equiv 1 \bmod n$.

**Pythagorean Triples** All relatively prime positive integer solutions to $x^2 + y^2 = z^2$ with $x$ odd and $y$ even are of the form $x = u^2 - v^2$ , $y = 2uv$, $z = u^2 + v^2$.

1. Let $p_n$ be the $n^{th}$ prime number. Show that the sequence $\{q_n\}$ defined by $q_n = p_{n+1} - p_n$ is unbounded.

**Solution:** Note that none of the numbers $n! + 2, n! + 3, \ldots, n! + n$ are prime. Thus there are arbitrarily large gaps in the sequence of primes.

2. Show that the product of four consecutive positive integers is never a perfect square.

**Solution:** Since $n(n+1)(n+2)(n+3) = (n^2+3n)(n^2+3n+2) = (n^2+3n+1)^2 - 1$, the product of four integers is one less than a perfect square. Since $n > 0$, the product is at least 24, but the only integer solutions to $x^2 - y^2 = 1$ are $(1, 0)$ and $(-1, 0)$. Thus the product is not a perfect square.

3. Find the last two digits of $3^{2011}$.

**Solution:** We are looking for a congruence modulo 100. We have that $\phi(100) = \phi(4) \cdot \phi(25) = 2(2-1)5(5-1) = 40$. Since $2011 = 50 \cdot 40 + 11$,

$$3^{2011} \equiv 3^{50 \cdot 40 + 11} \equiv (3^{50})^{40} 3^{11} \equiv 3^{11} \equiv (3^5)^2 \cdot 3 \equiv 43^2 \cdot 3 \equiv 49 \cdot 3 \equiv 47 (\bmod 100)$$

Thus the last two digits are 47.

4. How many positive integers divide at least one of $10^{40}$ and $20^{30}$?

**Solution:** This is (Putnam '83, A1). Answer: 2301. The factors of $10^{40}$ have the form $2^m 5^n$ with $0 \leq m, n \leq 40$. So there are $41^2 = 1681$ such factors. Factors of $20^{30} = 2^{60}5^{30}$ not dividing $10^{40}$ have the form $2^m 5^n$ with $41 \leq m \leq 60$ and $0 \leq n \leq 30$, so there are $20 \cdot 31 = 620$ such factors.

5. Show that for any positive integer $r$, we can find integers $a$, $b$ such that $a^2 - b^2 = r^3$.

**Solution:** This is (Putnam '54, B1). It suffices to take $a + b = r^2$ and $a - b = r$. Thus $a = \frac{r(r+1)}{2}$ and $b = \frac{r(r-1)}{2}$.

6. Find all solutions to $1! + 2! + 3! + \ldots + n! = m^2$ in positive integers.

**Solution:** First notice that $n!$ is a multiple of 10 for $n \geq 5$. Thus the last digit of $1! + 2! + 3! + \ldots + n!$ is 3 for $n \geq 5$. Since no perfect square ends in 3, it suffices to check $n \leq 4$. Thus the only solutions are $m = n = 3$ and $m = n = 1$.

7. Let $a > 1$. Show that $a^n + 1$ is prime only if $a$ is even and $n = 2^k$.

**Solution:** If $a$ is odd, $a^n + 1$ is an even number greater than 2. If $n = mq$, where $m$ is odd, then $a^q + 1 | a^n + 1$. (In general, $b + 1 | b^m + 1$ if $m$ is odd.) Thus, if $a^n + 1$ is prime for $a > 1$, then $a$ must be even, and $n$ should have no odd factor, i.e., $n$ is a power of 2.

8. Which members of the sequence 101, 10101, 1010101, ... are prime?

**Solution:** This is (Putnam '89, A1). Let $k_n$ represent the member of the sequence with $n$ 1's. It is obvious that 101 divides $k_{2n}$. So we need only consider $k_{2n+1}$.

But

$$k_{2n+1} = 1 + 10^2 + 10^4 + \ldots + 10^{4n} = \frac{10^{4n+2} - 1}{99} = \frac{10^{2n+1} + 1}{11} \cdot \frac{10^{2n+1} - 1}{9}.$$

Each of these is integral: the first is $1 - 10 + 10^2 - \ldots + 10^{2n}$, the second is 11...1 ($2n + 1$ digits 1's).

9. The number $2^{333}$ has 101 digits, and begins with 1. How many of the numbers in the set $2, 4, 8, 16, ..., 2^{333}$ begin with 4?

**Solution:** The solution is bases on the observation that between $10^k$ and $10^{k+1} - 1$, there is exactly one power of 2 starting with 1, exactly one power of two starting with 2 or 3, and exactly one power of 2 starting with a digit from $\{5, 6, 7, 8, 9\}$.