Concours Putnam

Atelier de Pratique
Le jeudi, 31 octobre 12h30-13h30
La salle 5448 Pav. André Aisenstad

Le principe du pigeonnier/de Dirichlet

If $n+1$ objects ("pigeons") are distributed among n boxes ("pigeon holes"), at least one of the boxes contains more than one object. More generally, if $kn+1$ objects are distributed among n boxes, at least one of the boxes contains more than k objects.

1. Show that among any five points inside an equilateral triangle of side length 1, there exist two points whose distance is at most 1/2.

> **Solution:** Divide the triangle into four congruent equilateral triangles of side length 1/2. Then use the pigeonhole principle to conclude that one of these must contain two points.

2. Given a set of 7 integers, show that there exist two of them whose difference or sum is divisible by 10.

> **Solution:** Split the remainders upon division by 10 into the 6 classes $\{0\}$, $\{1,9\}$, $\{2,8\}$, $\{3,7\}$, $\{4,6\}$, and $\{5\}$, and argue that if two integers fall into the same class then either their difference or their sum is congruent to 0 modulo 10.

3. Prove that from a set of ten distinct two-digit integers it is possible to select two disjoint non-empty subsets whose members have the same sum.

> **Solution:** The disjointness requirement is a red herring, since if two non- disjoint sets have the same sum, removing the common elements from each set leaves two disjoint sets that still have the same sum. Thus, we can ignore the disjointness requirement. The problem then becomes a relatively easy pigeonhole problem, with the possible values for the sums as pigeonholes (the values are among 10 and 945), and the sets (of which there are $2^{10}-1=1023$) as pigeons.

4. Show that any set $A \subset \{1,2,...,2n\}$ with at least $n+1$ elements contains two elements, one of which divides the other.

Solution: Write each element in A as $2^k a$ with a odd. There are n possible values for a, so if A has $n+1$ elements, two of these must have the same value for a. These two have the required divisibility property.

5. Several chess players are playing a single round robin tournament. Prove that at any moment of the tournament there are two chess players who have played the same number of matches by that moment.

Solution: Let there be n chess players. Then each could have played from 0 to $n-1$ games: a total of n options. It would seem that the Dirichlet principle does not work: we have n different chess players and n different numbers of games played.

Note, however, that if some chess player has not played a single game, then there is no chess player who has played all the games. That is, there cannot be a situation where there is a player who has played 0 games and a player who has played $n-1$ games. This means that there can be no more than $n-1$ different numbers of games played at any moment of the tournament (from 0 to $n-2$ or from 1 to $n-1$). According to the Dirichlet principle, at any moment of the tournament there are two players who have played the same number of games.

6. Let S be the set of real numbers of the form $a + b\sqrt{2}$, where a and b are integers. Show that S is dense on the real line, in the sense that, given any $\epsilon > 0$ and any real number x there exists an element $s \in S$ with $|s - x| < \epsilon$.

Solution: First, by considering pairs (a, b) with $0 \leq a, b \leq N$ for a large N, show that, for any ϵ, there exist two numbers of the given form whose difference is $\leq \epsilon$. Next, noting that the difference of two numbers of this form has again the same form, conclude that there exists a number of the required form with absolute value $\leq \epsilon$. Finally, by considering integer multiples of that number, show that given any x, there exists such a number with distance $\leq \epsilon$ of x.

7. Each point in the plane with integer coordinates is colored in one of n colors. Prove that there is a rectangle with vertices at points of the same color.

Solution: Consider a strip of $n+1$ consecutive horizontal rows of points. Consider the vertical rows of this strip consisting of $n+1$ points. There are only a finite number of ways to color $n+1$ points in n colors (the first point can be colored in n ways, independently of this the second point can also be colored in n ways, etc., in total there are n^{n+1} ways to color $n+1$ points in n colors). Therefore, among the vertical rows of the strip under consideration (there are infinitely many of these rows), there will be two identically colored rows A and B, i.e. such rows that the points of these rows located in the i-th horizontal row ($i = 1, 2, ..., n+1$) have the same color. Since there are $n+1$ integer points in the vertical row A, there will be two points X and Y of the same color in it. In the row B, two points X', Y', located in the same horizontal rows as the points X and Y, are colored the same color as X, Y (since the rows A and B are colored the same). Points X, Y, X', Y' are the vertices of a rectangle and have the same color, i.e. they form the desired quadruple of points.

8. The Fibonacci sequence is defined by $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Show that, given any positive integer k, there exists a Fibonacci number F_n ending in at least k zeros.

Solution: Fix k, and consider the pairs $(a_n, a_{n-1}) = (F_n \bmod 10^k, F_{n-1} \bmod 10^k)$. Note that, by the Fibonacci recurrence, a single such pair determines the entire sequence $\{a_n\}$ uniquely (forwards and backwards). Since there are only finitely many (namely, 10^{2k}) possible values for these pairs, the sequence (a_n, a_{n-1}) must be periodic. Since $(a_1, a_0) = (1, 1)$, there are infinitely many n with $(a_n, a_{n-1}) = (1, 1)$, and for each of these n, we have $a_{n-2} = 0$, by the Fibonacci recurrence.

9. Suppose $\mathcal{A}$ is a collection of subsets of $\{1, 2, \ldots, n\}$ with the property that any two sets in $\mathcal{A}$ have a non-empty intersection. Show that $\mathcal{A}$ has at most 2^{n-1} elements. Can the bound 2^{n-1} be lowered?

Solution: Split the 2^n subsets into pairs of the form $\{A, A^c\}$. Note that a set A with the given property can contain at most one element from each such pair. Since there are 2^{n-1} such pairs, $\mathcal{A}$ can have at most 2^{n-1} elements. The bound can not be lowered. Take $\mathcal{A}$ to be the collection of all the subsets that contain 1, there are 2^{n-1} such subsets.

10. Given any 5 distinct points on the surface of a sphere, show that we can find a closed hemisphere which contains at least 4 of them.

Solution: This is (Putnam, '02, A2). Take a great circle through two of the points. Then at least two of the other three points must lie in one of the hemispheres bounded by the great circle.

11. A partition of a set S is a collection of disjoint non-empty subsets (parts) whose union is S. For a partition π of $\{1, 2, \ldots, 9\}$, let $\pi(x)$ be the number of elements in the part containing x. Prove that for any two partitions π and π', there exist x, $y \in \{1, 2, \ldots, 9\}$, $x \neq y$, such that $\pi(x) = \pi(y)$ and $\pi'(x) = \pi'(y)$.

Solution: This is (Putnam '95, B1). Note that, for a given π, there can be at most 3 distinct values of $\pi(x)$ (since $1+2+3+4 = 10 > 9$). Thus, if $\pi(x) \geq 4$ for some x, two elements in the set containing x will share the same value of π' , and we are done. If not, given π and π' there are at most 9 possible values for the pair $(\pi(x), \pi'(x))$, namely $\{(i, j) : 1 \leq i, j \leq 3\}$. If one of these pairs occurs for more than one x, we are done. Otherwise, each of these pairs occurs exactly once. Therefore,

$$|\{x : \pi(x) = 1\}| = |\{x : \pi(x) = 2\}| = |\{x : \pi(x) = 3\}| = 3$$

But $\{x : \pi(x) = 2\}$ has an even number of elements, so this is impossible.