Concours Putnam

Atelier de Pratique

Le jeudi, 17 octobre 12h30-13h30 **Polynômes**

Factor Theorem. The polynomial $p(x) = a_n x^n + \ldots + a_1 x + a_0$ has a root α of multiplicity m, then $p(x) = (x-\alpha)^m q(x), q(\alpha) \neq 0$.

Elementary Symmetric Polynomials. Every symmetric polynomial in x_1, x_2,..., x_n can be expressed as a polynomial in $\sigma_1, \sigma_1, \ldots, \sigma_n$, where

$$\sigma_k = \sum_{1 \leq j_1 < j_2 < \ldots < j_k \leq n} x_{j_1} x_{j_2} \ldots x_{j_k}$$

Vieta's Formula. Let $z_1, z_2, \ldots z_n$ be the (possibly complex) roots of the monic polynomial $p(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$. Then $a_{n-k} = (-1)^k \sigma_k(z_1, z_2, \ldots, z_n)$ where σ_k is the elementary symmetric polynomial of degree k in n variables.

Identity Theorem. If $p(x)$ and $q(x)$ are polynomials of degree at most n, and $p(x_k) = q(x_k)$ for $1 \leq k \leq n+1$ for distinct $x_1, x_2, \ldots, x_{n+1}$, then $p(x) = q(x)$ for all x.

1. Let $\alpha = 2^{1/3} + 5^{1/2}$. Find a polynomial $p(x)$ with integer coefficients satisfying $p(\alpha) = 0$.

Solution: $(\alpha - \sqrt{5})^3 = 2 \Rightarrow \alpha^3 + 15\alpha - 2 = \sqrt{5}(3\alpha^2 + 5)$. Therefore, $(\alpha^3 + 15\alpha - 2)^2 - 5(3\alpha^2 + 5)^2 = 0$. Thus,

$$p(x) = (x^3 + 15x - 2)^2 - 5(3x^2 + 5)^2 = x^6 - 15x^4 - 4x^3 + 75x^2 - 60x - 121$$

has the required property.

2. Find a polynomial of degree at most 3 such that $p(2) = 3, p(3) = 5, p(5) = 8$ and $p(7) = 13$.

Solution: Let $p(x) = a + b(x-2) + c(x-2)(x-3) + d(x-2)(x-3)(x-5)$. Then $a = p(2) = 3; p(3) = a + b \Rightarrow b = 2; p(5) = a + 3b + 6c \Rightarrow c = -1/6; p(7) = a + 5b + 20c + 40d \Rightarrow d = -1/12$. Thus,

$$p(x) = 3 + 2(x-2) - \frac{(x-2)(x-3)}{6} - \frac{(x-2)(x-3)(x-5)}{12}.$$

3. If $x + y + z = 3, x^2 + y^2 + z^2 = 5, x^3 + y^3 + z^3 = 7$, find $x^4 + y^4 + z^4$.

Solution: Let $\sigma_1(x,y,z) = x+y+z$, $\sigma_2(x,y,z) = xy+yz+xz$ and $\sigma_3(x,y,z) = xyz$ denote the elementary symmetric polynomials in x, y, and z. We have, $\sigma_1 = 3$, $\sigma_1^2 - 2\sigma_2 = 5$ and $7 - 3\sigma_3 = \sigma_1(\sigma_1^2 - 3\sigma_2)$. Thus, $\sigma_2 = 2$ and $\sigma_3 = -2/3$. Now $x^4 + y^4 + z^4 = (x^2 + y^2 + z^2)^2 - 2(\sigma_2^2 - 2\sigma_1\sigma_3) = 9$.

4. Find all polynomials $P(x)$ satisfying $P(x^2+1) = (P(x))^2 + 1$ for all x and $P(0) = 0$.

Solution: Consider the sequence $\{u_k\}$ defined as follows: $u_0 = 0; u_k = u_{k-1}^2 + 1$ for $k \geq 1$. It can be easily proved by induction on k that $P(u_k) = u_k$ for all k. Since $u_k > u_{k-1}$ for all k, $P(x)$ coincides with x for infinitely many values. It follows from the Identity Theorem that $P(x) = x$.

5. Find a non-zero polynomial $P(x,y)$ such that $P([t],[2t]) = 0$ for all real numbers t.

Solution: This is (Putnam '05, B1). Answer: Let $[t] = n$. Thus $n \leq t < n+1$, i.e, $2n \leq 2t < 2n+2$. It follows that $[2t] = 2[t]$ or $[2t] = 2[t]+1$. Thus $P(x,y) = (y-2x)(y-2x-1)$ satisfies $P([t],[2t]) = 0$ for all t.

6. Suppose that the monic polynomial $p(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + 1$ has non-negative coefficients and n real roots. Show that $p(2) \geq 3^n$.

Solution: Let $y_1, y_2, \ldots y_n$ be the roots of $p(x)$. Since $p(x) \geq 1$ when $x \geq 0$, we have $y_i < 0$ for all i. Let $y_i' = -y_i$. Note that

$$p(2) = (2 + y_1')(2 + y_2') \ldots (2 + y_n').$$

By Vieta's formula, $y_1' y_2' \ldots y_n' = 1$. Also, by the AGM inequality, $2 + y_i' \geq 3(y_i')^{1/3}$. It follows that $p(2) \geq 3^n$.

Another solution. Again notice that the roots $y_i < 0$. Thus we have,

$$p(2) = \sum_{k=0}^{n} (-1)^k \sigma_k(y_1, \ldots, y_n) 2^{n-k} = \sum_{k=0}^{n} \sigma_k(|y_1|, \ldots, |y_n|) 2^{n-k}.$$

By AGM,

$$\frac{\sigma_k(|y_1|, \ldots, |y_n|)}{\binom{n}{k}} \geq \sqrt[\binom{n}{k}]{|y_1 \ldots y_n|^{\binom{n-1}{k-1}}} = 1.$$

Therefore,

$$p(2) \geq \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} = (1+2)^n = 3^n.$$

7. Let $p(x) = a_n x^n + \ldots + a_1 x + a_0$ be a polynomial with integer coefficients. If r is a rational root of $p(x)$, show that the numbers $a_n r$, $a_n r^2 + a_{n-1} r, \ldots, a_n r^n + a_{n-1} r^{n-1} + \ldots + a_1 r$ are all integers.

Solution: This (Putnam '04, B1). Let $r = b/c$, with $(b, c) = 1$ (i.e., b and c are relatively prime). Since $p(r) = 0$, we get, after clearing denominators,

$$a_n b^n + a_{n-1} b^{n-1} c + \ldots + a_0 c^n = 0.$$

For $1 \leq k \leq n$, define

$$p_k(b, c) = a_n b^n + a_{n-1} b^{n-1} c + \ldots + a_{n-k+1} b^{n-k+1} c^{k-1}$$

Note that $c^k | p_k(b, c)$. But $p_k(b, c) = b^{n-k}(a_n b^k + a_{n-1} b^{k-1} + \ldots + a_{n-k+1} b c^{k-1})$. Furthermore, $(b, c) = 1 \Rightarrow (b^{n-k}, c^k) = 1$. Thus,

$$c^k | a_n b^k + a_{n-1} b^{k-1} c + \ldots + a_{n-k+1} b c^{k-1}.$$

It follows that $a_n r^k + a_{n-1} r^{k-1} + \ldots + a_{n-k+1} r$ is an integer for $1 \leq k \leq n$.

8. Do there exist polynomials $a(x)$, $b(x)$, $c(y)$, $d(y)$ such that $1 + xy + x^2 y^2 = a(x)c(y) + b(x)d(y)$?

Solution: This is (Putnam '03, B1)

Suppose that such polynomials exist. Then $1 = c(0)a(x) + d(0)b(x)$, $1 + x + x^2 = c(1)a(x) + d(1)b(x)$, $1 - x + x^2 = c(-1)a(x) + d(-1)b(x)$. We have

$$\begin{aligned}
1 &= c(0)a(x) + d(0)b(x), \\
x &= \frac{c(1) - c(-1)}{2} a(x) + \frac{d(1) - d(-1)}{2} b(x), \\
x^2 &= \frac{c(1) + c(-1) - 2c(0)}{2} a(x) + \frac{d(1) + d(-1) - 2d(0)}{2} b(x).
\end{aligned}$$

So we have three linearly independent elements (the polynomials 1, x, x^2) in a subspace of dimension 2 (the vector space spanned by $a(x)$ and $b(x)$). Contradiction.

9. Is there a real polynomial of two variables that is positive, but can assume arbitrarily small values?

Solution: Example: $p(x, y) = (xy - 1)^2 + x^2$.