Concours Putnam

Atelier de Pratique

Le jeudi, 17 octobre 12h30-13h30 **Polynômes**

Factor Theorem. The polynomial $p(x) = a_n x^n + \ldots + a_1 x + a_0$ has a root α of multiplicity m, then $p(x) = (x - \alpha)^m q(x), q(\alpha) \neq 0$.

Elementary Symmetric Polynomials. Every symmetric polynomial in x_1, x_2,..., x_n can be expressed as a polynomial in $\sigma_1, \sigma_1, \ldots, \sigma_n$, where

$$\sigma_k = \sum_{1 \leq j_1 < j_2 < \ldots < j_k \leq n} x_{j_1} x_{j_2} \ldots x_{j_k}$$

Vieta's Formula. Let $z_1, z_2, \ldots z_n$ be the (possibly complex) roots of the monic polynomial $p(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$. Then $a_{n-k} = (-1)^k \sigma_k(z_1, z_2, \ldots, z_n)$ where σ_k is the elementary symmetric polynomial of degree k in n variables.

Identity Theorem. If $p(x)$ and $q(x)$ are polynomials of degree at most n, and $p(x_k) = q(x_k)$ for $1 \leq k \leq n+1$ for distinct $x_1, x_2, \ldots, x_{n+1}$, then $p(x) = q(x)$ for all x.

1. Let $\alpha = 2^{1/3} + 5^{1/2}$. Find a polynomial $p(x)$ with integer coefficients satisfying $p(\alpha) = 0$.

2. Find a polynomial of degree at most 3 such that $p(2) = 3, p(3) = 5, p(5) = 8$ and $p(7) = 13$.

3. If $x + y + z = 3, x^2 + y^2 + z^2 = 5, x^3 + y^3 + z^3 = 7$, find $x^4 + y^4 + z^4$.

4. Find all polynomials $P(x)$ satisfying $P(x^2 + 1) = (P(x))^2 + 1$ for all x and $P(0) = 0$.

5. Find a non-zero polynomial $P(x, y)$ such that $P([t], [2t]) = 0$ for all real numbers t.

6. Suppose that the monic polynomial $p(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + 1$ has non-negative coefficients and n real roots. Show that $p(2) \geq 3^n$.

7. Let $p(x) = a_n x^n + \ldots + a_1 x + a_0$ be a polynomial with integer coefficients. If r is a rational root of $p(x)$, show that the numbers $a_n r$, $a_n r^2 + a_{n-1} r, \ldots, a_n r^n + a_{n-1} r^{n-1} + \ldots + a_1 r$ are all integers.

8. Do there exist polynomials $a(x)$, $b(x)$, $c(y)$, $d(y)$ such that $1 + xy + x^2y^2 = a(x)c(y) + b(x)d(y)$?

9. Is there a real polynomial of two variables that is positive, but can assume arbitrarily small values?