Concours Putnam

Atelier de Pratique
Le jeudi, 3 octobre 12h30-13h30
5448 Pav. André Aisenstadt

Suites et Séries

Geometric series

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad |x| < 1$$

Finite geometric series

$$\sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x} \quad n = 1, 2, \ldots, x \neq 1$$

Exponential series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

Logarithmic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = \log(1+x) \quad -1 < x \leq 1$$

1. Compute the following
    1. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$
    2. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$
    3. $\sum_{n=k}^{\infty} x^n$
    4. $\sum_{n=0}^{\infty} \frac{n}{2^n}$
    5. $\sum_{n=1}^{\infty} \frac{1}{n2^n}$
    6. $\sum_{n=1}^{\infty} \frac{1}{n(n+3)}$
    7. $\sum_{n=0}^{\infty} \frac{(n+1)^2}{n!}$
    8. $\sum_{n=0}^{\infty} \binom{n+k}{k} x^n$ for $k = 0, 1, 2, ... |x| < 1$.
2. Evaluate

$$\sum_{n=1}^{\infty} \frac{(-1)^{[2^n x]}}{2^n} \quad 0 < x < 1,$$

where $[t]$ denotes the greatest integer $\leq t$.

3. Evaluate
$$\sum_{n=1}^{\infty} \frac{s(n)}{n(n+1)},$$
where $s(n)$ is the number of 1's in the binary expansion of n.

4. Let a and d be positive integers. Show that the arithmetic progression $a, a+d, a+2d, \ldots$ either contains no perfect square or contains infinitely many perfect squares.

5. Solve: $x_{n+1} = 2x_n(1 - x_n)$, with $x_1 = -1$.

6. Let $\{x_n\}$ be a sequence of real numbers satisfying $x_n = (x_{n-1} + x_{n-2})/2$. Show that the sequence converges, and find the limit in terms of x_0 and x_1.

7. Let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be infinite sequences of positive integers. Show that there exist distinct indices p and q such that $x_p \geq x_q$, $y_p \geq y_q$, and $z_p \geq z_q$.

8. Let $a_1 = a_2 = 1$ and $a_n = \frac{a_{n-1}^2+2}{a_{n-2}}$ for $n \geq 3$. Show that a_n is an integer for all n.

9. For each integer $n \geq 0$, let $d(n) = n - m^2$, where m is the largest integer with $m^2 \leq n$. Define a sequence $\{b_k\}$ by $b_0 = B$; $b_{k+1} = b_k + d(b_k)$. For what positive integers B is $\{b_k\}$ eventually constant?

10. A sequence of natural numbers is given by $x_1 = 2$, $x_{n+1} = \lfloor 1.5x_n \rfloor$. Prove that it contains infinitely many odd numbers, and infinitely many even numbers