# Concours Putnam

Atelier de Pratique

Le mardi, 21 novembre 12h30-13h30

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**Récurrences**

1. Let $a_0 = 1$, $a_1 = \frac{3}{5}$, $a_{n+1} = \frac{6}{5}a_n - a_{n-1}$. Show that $|a_n| \leq 1$ for all $n$.

**Solution:** The characteristic equation is

$$x^2 - \frac{6}{5}x + 1 = 0$$

with roots $\frac{3\pm 4i}{5}$. The solution is given by $a_n = \alpha\left(\frac{3+4i}{5}\right)^n + \beta\left(\frac{3-4i}{5}\right)^n$. Then $\alpha+\beta = 1$ and $\alpha = \beta = \frac{1}{2}$. Thus $a_n = \frac{1}{2}2\operatorname{Re}\left(\frac{3+4i}{5}\right)^n$. Then $|a_n| \leq \left|\frac{3+4i}{5}\right|^n = 1$.

2. Solve $a_{n+1} = \sqrt{a_n a_{n-1}}$ where $0 < a_0 < a_1$ and find $\lim_{n\to\infty} a_n$.

**Solution:** It is not hard to see that $a_n \neq 0$ from the recurrence. We will prove that $a_n = a_1^{b_n} a_0^{1-b_n}$. We have $b_0 = 0$, $b_1 = 1$, and then $a_{n+1} = \sqrt{a_n a_{n-1}} = a_1^{\frac{b_n+b_{n-1}}{2}} a_0^{1-\frac{b_n+b_{n-1}}{2}}$. We get

$$2b_{n+1} = b_n + b_{n-1}.$$

The characteristic equation is $2x^2 - x - 1 = 0$. Then $b_n = \alpha + \beta(-1/2)^n$ with $\alpha + \beta = 0$ and $2\alpha - \beta = 2$. So $\alpha = \frac{2}{3}$ and $\beta = -\frac{2}{3}$. This means that

$$b_n = \frac{2}{3}\left(1 - \left(-\frac{1}{2}\right)^n\right) \to \frac{2}{3}$$

Then

$$\lim_{n\to\infty} a_n = a_1^{2/3} a_0^{1/3}.$$

Alternatively, we can write $b_n = \log a_n$ and solve the recurrence $b_{n+1} = \frac{b_n + b_{n+1}}{2}$.

3. Prove that the sequence $a_0 = 2, a_1 = 3, a_2 = 6, a_3 = 14, a_4 = 40, a_5 = 152, a_6 = 784$, ... with general term $a_n = (n+4)a_{n-1} - 4na_{n-2} + (4n-8)a_{n-3}$ is the sum of two well-known sequences.

**Solution:** This is (Putnam 1990, A1). The answer is $n! + 2^n$. $a_5, a_6$ remind of 120, 720. We check:

$$\begin{aligned} & (n+4)a_{n-1} - 4na_{n-2} + (4n-8)a_{n-3} \\ = \; & (n+4)((n-1)! + 2^{n-1}) - 4n((n-2)! + 2^{n-2}) + 4(n-2)((n-3)! + 2^{n-3}) \\ = \; & (n+4)(n+1)! + (4-4n)(n-2)! + 2^{n-1}(n+4-2n+n-2) \\ = \; & n! + 2^n \end{aligned}$$

4. The sequence $a_n$ of non-zero reals satisfies $a_n^2 - a_{n-1}a_{n+1} = 1$ for $n \geq 1$. Prove that there exists a real number $\alpha$ such that $a_{n+1} = \alpha a_n - a_{n-1}$ for $n \geq 1$.

**Solution:** This is (Putnam 1993, A2). For $n \geq 2$ define $b_n = \frac{a_n + a_{n-2}}{a_{n-1}}$. Then the relation given shows that $b_{n+1} = b_n = \ldots = b_2 = \alpha$.

5. Find

$$\lim_{n\to\infty} (2+\sqrt{2})^n - \lfloor (2+\sqrt{2})^n \rfloor$$

where $\lfloor x \rfloor$ is the largest integers $\leq x$.

**Solution:** Notice that $(2+\sqrt{2})^n + (2-\sqrt{2})^n$ is an integer. Since

$$0 < (2+\sqrt{2})^n - \lfloor (2+\sqrt{2})^n \rfloor + (2-\sqrt{2})^n - \lfloor (2-\sqrt{2})^n \rfloor < 2$$

and it is the difference of two integers, we conclude that it is equal to 1. Now $\lim_{n\to\infty}(2-\sqrt{2})^n = 0$ implies that $\lim_{n\to\infty}(2-\sqrt{2})^n - \lfloor (2-\sqrt{2})^n \rfloor = 0$. Then

$$\lim_{n\to\infty} (2+\sqrt{2})^n - \lfloor (2+\sqrt{2})^n \rfloor = 1.$$

6. Solve

$$f(n+1) = 1 + \sum_{i=0}^{n-1} f(i)$$

with $f(0) = 1$.

**Solution:** Notice that $f(n+2) - f(n+1) = f(n)$, and that $f(1) = 1 + \sum_{i=0}^{-1} = 1$. We get $f(n) = F_{n+1}$ the Fibonacci sequence.

7. Solve
$$y_n(1 + ay_{n-1}) = 1.$$

**Solution:** Write $y_n = \frac{x_n}{x_{n+1}}$. Then
$$x_{n+1} = x_n + ax_{n-1}$$
and then continue as always.

8. Given $a_n = (n^2 + 1)3^n$, find a recurrence relation $a_n + pa_{n+1} + qa_{n+2} + ra_{n+3} = 0$. Hence evaluate $\sum_{n=0}^{\infty} a_n x^n$.

**Solution:** This is (Putnam 1939, B3). We look for a relation between $b_n = \frac{a_n}{3^n}$, because that takes care of the powers of 3. So, ignoring the $3^n$, we are looking at: $n^2 + 1$, $n^2 + 2n + 2$, $n^2 + 4n + 5$, $n^2 + 6n + 10$.

We try to get a linear combination of the first three which is constant. Subtracting twice the second from the third gets rid of the $n$ term, then adding the first gets rid of the $n^2$ term. So, $b_{n+2} - 2b_{n+1} + b_n = 2$. But $b_{n+3} - 2b_{n+2} + b_{n+1}$ has the same value, so subtracting:
$$a_{n+3} - 9a_{n+2} + 27a_{n+1} - 27a_n = 0,$$
which is the required recurrence relation.

Let the power series sum to $y$. Then taking $y - 9xy + 27x^2y - 27x^3y$ will give $a_{n+3} - 9a_{n+2} + 27a_{n+1} - 27a_n$ as the coefficient of $x^{n+3}$, so we need only worry about the early terms: $a_0 + (a_1 - 9a_0)x + (a_2 - 9a_1 + 27a_0)x^2 = (1 - 3x + 18x^2)$. Hence $y = \frac{1-3x+18x^2}{1-9x+27x^2-27x^3}$.

9. The sequence $a_n$ is defined by $a_1 = 2, a_{n+1} = a_n^2 - a_n + 1$. Show that any pair of values in the sequence are relatively prime and that $\sum \frac{1}{a_n} = 1$.

**Solution:** This is (Putnam 1956, B6). We show by induction on $k$ that $a_{n+k} \equiv 1(\bmod a_n)$. Obviously true for $k = 1$. Suppose it is true for $k$. Then for some $m$, $a_{n+k} = ma_n + 1$. Hence $a_{n+k+1} = a_{n+k}(ma_n + 1 - 1) + 1 = a_{n+k}ma_n + 1 \equiv 1(\bmod a_n)$. So the result is true for all $k$. Hence any pair of distinct $a_n$ are relatively prime.

We show by induction that $\sum_1^n \frac{1}{a_r} = 1 - \frac{1}{a_{n+1}-1}$. For $n = 2$, this reduces to $\frac{1}{2} = 1 - \frac{1}{3-1}$, which is true. Suppose it is true for $n$. Then $\sum_1^{n+1} \frac{1}{a_r} = 1 - k$, where

$k = \frac{1}{a_{n+1}-1} - \frac{1}{a_{n+1}} = \frac{1}{a_{n+1}^2 - a_{n+1}} = \frac{1}{a_{n+2}-1}$, so it is true for $n+1$. But $a_n \to \infty$, so $\sum \frac{1}{a_n} = 1$.