Université de Montréal *Automne 2023*

Concours Putnam

Atelier de Pratique
Le mardi, 14 novembre 12h30-13h30
5448 Pav. André Aisenstadt
Théorie des nombres

1. Show that the sum of two consecutive primes is never twice a prime.

> **Solution:** If p and q are consecutive primes and $p+q=2r$, then $r=\frac{p+q}{2}$ and $p<r<q$, but there are no primes between p and q.

2. Prove that two consecutive Fibonacci numbers are always relatively prime.

> **Solution:** This can be proved by induction. Base case: $F_1 = 1$ and $F_2 = 1$ are in fact relatively prime. Induction Step: we must prove that if F_n and F_{n+1} are relatively prime then so are F_{n+1} and F_{n+2}. But this follows from the recursive definition of the Fibonacci sequence: $F_n + F_{n+1} = F_{n+2}$; any common factor of F_{n+1} and F_{n+2} would be also a factor of F_n, and consequently it would be a common factor of F_n and F_{n+1} (which by induction hypothesis are relatively prime.)

3. Show that there exist 1999 consecutive numbers, each of which is divisible by the cube of an integer.

> **Solution:** Pick 1999 different prime numbers $p_1, p_2, \ldots, p_{1999}$ (we can do that because the set of prime numbers is infinite) and solve the following system of 1999 congruences:
>
> $$\begin{cases} x \equiv 0 \pmod{p_1^3} \\ x \equiv -1 \pmod{p_2^3} \\ x \equiv -2 \pmod{p_3^3} \\ \quad \ldots \\ x \equiv -1998 \pmod{p_{1999}^3} \end{cases}$$
>
> According to the Chinese Remainder Theorem, that system of congruences has a solution $x(\bmod M)$ with $M = p_1^3 \ldots p_{1999}^3$. For $k = 1, \ldots, 1999$ we have that $x + k \equiv 0(\bmod p_{k+1}^3)$, hence $x+k$ is in fact a multiple of p_{k+1}^3.

4. Find all non-negative integral solutions $(n_1, n_2, \ldots, n_{14})$ to

$$n_1^4 + n_2^4 + \ldots + n_{14}^4 = 1599.$$

Solution: This is USAMO, 1979. We look at the equation modulo 16. First we notice that $n^4 \equiv 0$ or $1(\mod 16)$ depending on whether n is even or odd. On the other hand $1599 \equiv 15(\mod 16)$. So the equation can be satisfied only if the number of odd terms in the LHS is 15 modulo 16, but that is impossible because there are only 14 terms in the LHS. Hence the equation has no solution.

5. a) Do there exist 2 irrational numbers a and b greater than 1 such that $\lfloor a^m \rfloor \neq \lfloor b^n \rfloor$ for every positive integers m, n?

   b) Do there exist 2 irrational numbers a and b greater than 1 such that $\lfloor am \rfloor \neq \lfloor bn \rfloor$ for every positive integers m, n?

Solution: a) The answer is yes. Let $a = \sqrt{6}$ and $b = \sqrt{3}$. Assume $\lfloor a^m \rfloor = \lfloor b^n \rfloor = k$ for some positive integers m, n. Then, $k^2 \leq 6^m < (k+1)^2 = k^2 + 2k + 1$, and $k^2 \leq 3^n < (k+1)^2 = k^2 + 2k + 1$. Hence, subtracting the inequalities and taking into account that $n > m$:

$$2k \geq |6^m - 3^n| = 3^m |2^m - 3^{n-m}| \geq 3^m.$$

Hence $\frac{9^m}{4} \leq k^2 \leq 6^m$, which implies $\frac{1}{4} \leq \left(\frac{2}{3}\right)^m$. This holds only for $m = 1, 2, 3$. These values of m can be ruled out by checking the values of

$$\lfloor a \rfloor = 2, \lfloor a^2 \rfloor = 6, \lfloor a^3 \rfloor = 14,$$

$$\lfloor b \rfloor = 1, \lfloor b^2 \rfloor = 3, \lfloor b^3 \rfloor = 5, \lfloor b^4 \rfloor = 9, \lfloor b^5 \rfloor = 15.$$

Hence, $\lfloor am \rfloor \neq \lfloor bn \rfloor$ for every positive integers m, n.

6. Suppose $n > 1$ is an integer. Show that $n^4 + 4^n$ is not prime.

Solution: If n is even then $n^4 + 4^n$ is even and greater than 2, so it cannot be prime. If n is odd, then $n = 2k + 1$ for some integer k, hence $n^4 + 4^n = n^4 + 4(2^k)^4$. Next, use Sophie Germain's identity: $a^4 + 4b^4 = (a^2 + 2b^2 + 2ab)(a^2 + 2b^2 - 2ab)$.

7. Prove that there are no primes in the following infinite sequence of numbers:

$$1001, 1001001, 1001001001, 1001001001001, \ldots$$

Solution: Each of the given numbers can be written

$$1 + 1000 + 1000^2 + \ldots + 1000^n = p_n(10^3)$$

where $p_n(x) = 1 + x + x^2 + \ldots + x^n$, $n = 1, 2, 3, \ldots$ We have $(x-1)p_n(x) = x^{n+1} - 1$. If we set $x = 10^3$, we get:

$$999p_n(10^3) = 10^{3(n+1)} - 1 = (10^{n+1} - 1)(10^{2(n+1)} + 10^{n+1} + 1).$$

If $p_n(10^3)$ were prime it should divide one of the factors on the RHS. It cannot divide $10^{n+1} - 1$, because this factor is less than $p_n(10^3)$, so $p_n(10^3)$ must divide the other factor. Hence $10^{n+1} - 1$ must divide 999, but this is impossible for $n > 2$. It only remains to check the cases $n = 1$ and $n = 2$. But $1001 = 7 \cdot 11 \cdot 13$, and $1001001 = 3 \cdot 333667$, so they are not prime either.

8. Let n be a positive integer. Suppose that 2^n and 5^n begin with the same digit. What is the digit?

Solution: The answer is 3. Note that $2^5 = 32, 5^5 = 3125$, so 3 is in fact a solution. We will prove that it is the only solution. Let d be the common digit at the beginning of 2^n and 5^n. Then

$$d10^r \leq 2^n < (d+1)10^r,$$

$$d10^s \leq 5^n < (d+1)10^s$$

for some integers r, s. Multiplying the inequalities we get

$$d^2 10^{r+s} \leq 10^n < (d+1)^2 10^{r+s},$$

$$d^2 \leq 10^{n-r-s} < (d+1)^2,$$

so d is such that between d^2 and $(d+1)^2$ there must be a power of 10. The only possible solutions are $d = 1$ and $d = 3$. The case $d = 1$ can be ruled out because that would imply $n = r + s$, and from the inequalities above would get

$$5^r \leq 2^s < 2 \cdot 5^r,$$

$$2^s \leq 5^r < 2 \cdot 2^s,$$

hence $2^s = 5^r$, which is impossible unless $r = s = 0$ (implying $n = 0$). Hence, the only possibility is $d = 3$.

9. Prove that if n is an integer greater than 1, then n does not divide $2^n - 1$.

Solution:

Let p the smallest prime divisor of n. Then $n|2^n - 1$ implies $p|2^n - 1$. Thus we have $2^n \equiv 1(\mod p)$ and we also have $2^{p-1} \equiv 1(\mod p)$ by Fermat's little theorem. Also note that p must be odd, since n is odd. We use that if $2^a \equiv 2^b \equiv 1(\mod p)$, then $2^{\gcd(a,b)} \equiv 1(\mod p)$. Let $d = \gcd(n, p-1)$. Then $2^d \equiv 1(\mod p)$. But p is the smallest prime divisor of n and all prime divisors of $p-1$ are less than p, we conclude that n and $p-1$ do not have common prime divisors so $d = 1$, and we get a contradiction.