Concours Putnam

Atelier de Pratique
Le mardi, 7 novembre 12h30-13h30
5448 Pav. André Aisenstadt
Géométrie

1. Let A, B, C, D be four points in space forming a quadrilateral. Show that the midpoints of AB, BC, CD, DA form a parallellogram.

Solution: Consider A, B, C, D as vectors in $\mathbb{R}^n$. Then the midpoints are given by $P = (A+B)/2$, $Q = (B+C)/2$, $S = (A+D)/2$, $R = (D+C)/2$. Then $Q - P = (C-A)/2 = R - S$, $P - S = (B-D)/2 = Q - R$. So $PQRS$ is a parallelogram.

2. Let A, B, C, D be four points on a plane. Prove that $|AB||CD| + |BC||AD| \geq |AC||BD|$.

Solution: Think of A, B, C, D as of four complex numbers. We know by algebraic manipulation with the complex multiplication that $(A-B)(C-D)+(B-C)(A-D) = (A-C)(B-D)$. Hence the inequality follows by taking norms and using their multiplicativity and the triangle inequality.

3. Let $v_1, \ldots, v_k$ be k vectors in the plane of norm $|v_i| \leq 1$. Prove that there exists a choice of signs such that $v = \pm v_1 \ldots \pm v_k$ satisfies $|v| \leq \sqrt{2}$.

Solution: We can suppose that no vectors vanish, by discarding all zero vectors. The question is trivial for $k = 1$. We first solve the question for $k = 2$. If a, b are vectors of norm at most one, then by switching a to $-a$ if necessary we can assume that $\langle a, b\rangle \leq 0$. Then $|a+b|^2 = |a|^2 + |b|^2 + 2\langle a, b\rangle \leq 2$, so $|a+b| \leq \sqrt{2}$. We will now reduce the question to $k = 2$ by showing that for each three vectors a, b, c with norm at most 1, at least one of the six vectors $a \pm b, b \pm c, a \pm c$ is of norm at most one (which we apply recursively). Indeed, at least one pair x, y of the six vectors $\{\pm a, \pm b, \pm c\}$ is at angle $\theta \geq 2\pi/3$, so $\langle x, y\rangle = |x||y|\cos(\theta) \leq -|x||y|/2$ and hence $|x+y|^2 = |x|^2 + |y|^2 + 2\langle x, y\rangle \leq |x|^2 + |y|^2 - |x||y| \leq \max\{|x|^2, |y|^2\} \leq 1$.

4. Let $P_1, \ldots, P_n$ be points on the unit sphere. Prove that $\sum_{i \leq j} |P_i P_j|^2 \leq n^2$.

Solution: Thinking of these points as vectors, recall that $|v|^2 = \langle v, v\rangle$ for all $v \in \mathbb{R}^k$, where $\langle v, w\rangle = v_1w_1 + \ldots + v_kw_k$ is the scalar product. Now we know that $|P_i| = 1$ for all i, and $|P_iP_j|^2 = \langle P_i - P_j, P_i - P_j\rangle$, hence

$$|P_iP_j|^2 = |P_i|^2 + |P_j|^2 - 2\langle P_i, P_j\rangle = 2 - 2\langle P_i, P_j\rangle.$$

Hence by summing over all $i < j$ (since $|P_iP_i| = 0$ for all i), we obtain

$$\sum_{i\leq j} |P_iP_j|^2 = 2n(n-1)/2 - 2\sum_{i<j}\langle P_i, P_j\rangle.$$

On the other hand we know that

$$0 \leq \langle \sum_i P_i, \sum_j P_j\rangle = \sum_i |P_i|^2 + 2\sum_{i<j}\langle P_i, P_j\rangle.$$

Hence $-2\sum_{i<j}\langle P_i, P_j\rangle \leq \sum_i |P_i|^2 = n$, so we conclude that

$$\sum_{i\leq j} |P_iP_j|^2 \leq n^2 - n + n = n^2$$

5. Does the circle $S^1 = \{x^2 + y^2 = 1\}$ in the plane contain a closed subset that contains exactly one of each pair of diametrically opposite points?

Solution: (This is Putnam 1975, B4) The map $\tau : v = (x, y) \mapsto -v = (-x, -y)$ is a homeomorphism from S^1 to S^1, a continuous mapping whose inverse is also continuous, and hence takes closed sets to closed sets. Hence K and $\tau(K)$ are disjoint closed sets whose union is S^1, but that is impossible since S^1 is connected. (For example, take a continuous arc $\gamma : [0, 1] \to \mathbb{R}$ joining $\gamma(0) = x \in K$ and $\gamma(1) = y \in \tau(K)$. Then for the point $t_0 = \inf\{t \mid \gamma(t) \in \tau(K)\}$ we have $\gamma(t_0) \in \tau(K)$ because $\tau(K)$ is closed and γ is continuous, but also $\gamma(t_0) \in K$ because we can find $t_j \to t_0$, $t_j \leq t_0$, $\gamma(t_j) \in K$, and K is closed and γ is contiuous. But then K, $\tau(K)$ are not disjoint, contradiction.)

6. Show that there are no 7 lines in the plane such that there are at least 6 points in the plane which lie on the intersection of just three of the lines and at least 4 points in the plane which lie on the intersection of just two of the lines.

Solution: (This is Putnam 1973, A6) From 7 lines we can choose just 21 pairs. An intersection of 3 lines accounts for 3 distinct pairs of lines, an intersection of 2 lines for 1 pair. Hence the configuration given would have at least $6 \cdot 3 + 4 \cdot 2 = 22$ distinct pairs of lines.