Concours Putnam

Atelier de Pratique
Le mardi, 31 octobre 12h30-13h30
5448 Pav. André Aisenstadt

Suites et séries

1. Let u be a real number with $0 < u < 1$. Let $u_0 = u$, and for $n \geq 1$ define u_n recursively by
$$u_n = \frac{1}{u_{n-1}} + u.$$
Prove that the sequence $\{u_n\}_{n\geq 1}$ converges and find its limit.

2. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence such that $\lim_{n\to\infty}(x_n - x_{n-2}) = 0$. Show that
$$\lim_{n\to\infty} \frac{x_n - x_{n-1}}{n} = 0.$$

3. Does the series
$$\sum_{n=0}^{\infty} \frac{n^n}{2^{n^2}}$$
converge?

4. Decide if the series
$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$
converges

5. Let a_n be a sequence of positive reals satisfying $a_n \leq a_{2n} + a_{2n+1}$ for all n. Prove that $\sum a_n$ diverges.

6. The sequence a_n is monotonic and $\sum a_n$ converges. Show that $\sum n(a_n - a_{n+1})$ converges.

7. Does $\sum_{n\geq 0} \frac{n!k^n}{(n+1)^n}$ converge or diverge for $k = \frac{19}{7}$?

8. The real sequence a_n satisfies $a_n = \sum_{k=n+1}^{\infty} a_k^2$. Show $\sum a_n$ does not converge unless all a_n are zero.

9. The series $\sum a_n$ of non-negative terms converges and $a_i \leq 100a_n$ for $i = n, n+1, n+2, \ldots, 2n$. Show that $\lim_{n\to\infty} na_n = 0$.