Concours Putnam

Atelier de Pratique
Le mardi, 31 octobre 12h30-13h30
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Suites et séries

1. Let u be a real number with $0 < u < 1$. Let $u_0 = u$, and for $n \geq 1$ define u_n recursively by

$$u_n = \frac{1}{u_{n-1}} + u.$$

Prove that the sequence $\{u_n\}_{n\geq 1}$ converges and find its limit.

Solution: Assuming the existence of the limit $L = L(u)$, substitute L for u_n and u_{n-1} in the equation and solve. It gives a quadratic equation with two roots, one positive and one negative. The recurrence forces u_n to be positive, hence L is positive and equal to $L = \frac{u+\sqrt{u^2+4}}{2}$. To prove the existence of the limit let $d_n = u_n - L$. The recurrence for u_n and the equation defining L gives $d_n = -\frac{d_{n-1}}{Lu_{n-1}}$. Iterating this recurrence, $d_n = \frac{d_{n-2}}{L^2 u_{n-1}u_{n-2}}$. Since $L > 1$ and $u_{n-1}u_{n-2} > 1$ (which follows from the recurrence for u_{n-1}), this implies that d_n tends to 0 at a geometric rate and thus $u_n \to L$ as $n \to \infty$.

2. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence such that $\lim_{n\to\infty}(x_n - x_{n-2}) = 0$. Show that

$$\lim_{n\to\infty} \frac{x_n - x_{n-1}}{n} = 0.$$

Solution: This is (Putnam 1970 A4). Let $b_n = x_n - x_{n-2}$. Then there is an $m > 2$ such that $|b_n| < \frac{\varepsilon}{2}$ for $n > m$. Then

$$\begin{aligned} x_n - x_{n-1} &= (x_n - x_{n-2}) - (x_{n-1} - x_{n-3}) + (x_{n-2} - x_{n-4}) - \dots \\ &= \sum_{i=2}^{n} (-1)^{n-i} b_i + (-1)^{n+1}(x_1 - x_0) \\ &= \left(\sum_{2\leq i\leq m} (-1)^{n-i} b_i + (-1)^{n+1}(x_1 - x_0) \right) + \sum_{m<i\leq n} (-1)^{n-i} b_i. \end{aligned}$$

Now

$$\left| \sum_{2\leq i\leq m} (-1)^{n-i} b_i + (-1)^{n+1}(x_1 - x_0) \right| \leq \sum_{2\leq i\leq m} |b_i| + |x_1 - x_0| = C_m$$

For $n > m'$ $\frac{C_m}{n} < \frac{\varepsilon}{2}$. On the other hand,

$$\left|\sum_{m<i\leq n}(-1)^{n-i}b_i\right| \leq \sum_{m<i\leq n}|b_i| \leq \frac{\varepsilon n}{2}$$

Thus, for $n \geq m, m'$,

$$\frac{x_n - x_{n-1}}{n} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \epsilon,$$

which is what we wanted to prove.

3. Does the series

$$\sum_{n=0}^{\infty}\frac{n^n}{2^{n^2}}$$

converge?

Solution: It converges by Cauchy's criterion:

$$\lim_{n\to\infty}\left(\frac{n^n}{2^{n^2}}\right)^{\frac{1}{n}} = \lim_{n\to\infty}\frac{n}{2^n} = 0.$$

4. Decide if the series

$$\sum_{n=2}^{\infty}\frac{1}{n\ln n}$$

converges

Solution: It suffices to compare with the function $\frac{1}{x\ln x}$. The sum has to be compared with the integral

$$\int_2^{\infty}\frac{dx}{x\ln x} = \ln(\ln x)|_2^{\infty}$$

but the limit does not exist, so the original series diverges.

5. Let a_n be a sequence of positive reals satisfying $a_n \leq a_{2n} + a_{2n+1}$ for all n. Prove that $\sum a_n$ diverges.

Solution: This is (Putnam 1994, A1)

$$\sum a_n = a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + \dots$$

and each bracket as sum at least $a_1 > 0$.

6. The sequence a_n is monotonic and $\sum a_n$ converges. Show that $\sum n(a_n - a_{n+1})$ converges.

Solution: (Putnam 1952, B5) The sum of the first n terms is

$$(a_1 - a_2) + 2(a_2 - a_3) + \dots + n(a_n - a_{n+1}) = a_1 + a_2 + a_3 + \dots + a_n - na_{n+1}.$$

We are given that $\sum a_n$ converges, so it is sufficient to show that the sequence na_{n+1} converges to zero.

But since $\sum a_n$ converges, $|a_{n+1} + a_{n+2} + \dots + a_{2n}|$ is arbitrarily small for n sufficiently large. Since a_n is monotonic, this implies that na_{n+1} is arbitrarily small for n sufficiently large.

7. Does $\sum_{n\geq 0} \frac{n!k^n}{(n+1)^n}$ converge or diverge for $k = \frac{19}{7}$?

Solution: This is (Putnam 1942, A3). The nth term divided by the $n-1$th term is $\frac{knn^{n-1}}{(n+1)^n} = \frac{k}{(1+1/n)^n}$ which tends to $\frac{k}{e}$. But $\frac{k}{e} < 1$, so the series converges by the ratio test.

8. The real sequence a_n satisfies $a_n = \sum_{k=n+1}^{\infty} a_k^2$. Show $\sum a_n$ does not converge unless all a_n are zero.

Solution: This is (Putnam 1954, A6). Clearly $a_n \geq 0$. If any $a_n = 0$, then all subsequent a_i must be zero, and, by a trivial induction, all previous a_i. So assume no $a_n = 0$.

Notice that $a_{n-1} = a_n^2 + a_n$. But we have that $a_n^2 > 0$, so $a_{n-1} > a_n$.

If the sum converges, then we can take n sufficiently large that $a_{n+1} + a_{n+2} + a_{n+3} + \dots < 1$. Then $a_n = a_{n+1}^2 + a_{n+2}^2 + a_{n+3}^2 + \dots < a_n(a_{n+1} + a_{n+2} + a_{n+3} + \dots) < a_n$. Contradiction. So the sum does not converge.

9. The series $\sum a_n$ of non-negative terms converges and $a_i \leq 100a_n$ for $i = n, n+1, n+2, \ldots, 2n$. Show that $\lim_{n\to\infty} na_n = 0$.

Solution: This is (Putnam 1963, B5). We need to invert the inequality given. We are given a collection of a_i which are less than a fixed a_n. We want to fix a_n and find a collection of a_j such that $a_n \leq 100a_j$.

Evidently, $a_{2n} \leq 100a_{2n-1}, a_{2n} \leq 100a_{2n-2}, a_{2n} \leq 100a_{2n-3}, \ldots, a_{2n} \leq 100a_n$. Adding and multiplying by two, $2na_{2n} \leq 200(a_n + a_{n+1} + \ldots + a_{2n-1})$. But $\sum a_n$ converges, so $(a_n + a_{n+1} + \ldots + a_{2n-1}) < \frac{\varepsilon}{200}$ for all sufficiently large n, and hence $2na_{2n} < \varepsilon$ for sufficiently large n. Similarly for $(2n+1)a_{2n+1}$. So na_n tends to zero.