Concours Putnam

Atelier de Pratique
Le mardi, 10 octobre 12h30-13h30
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Fonctions

1. The functions $f(x) = 4x - 4x^2$ and $\sin \pi x$ agree at $x = 0, 1/2$, and 1. Show that $f(x) \geq \sin \pi x$ for $0 \leq x \leq 1$.

2. Determine, with proof, all functions f defined on the set of integers and satisfying

$$f(n+m) + f(n-m) = 2(f(m) + f(n))$$

for all n and m.

3. Let $f(x) = \frac{x^3 e^{x^2}}{(1-x^2)^2}$. Find $f^{(2012)}(0)$. (Here $f^{(n)}$ denotes the nth derivative of f.)

4. Let

$$f(x) = \frac{1}{1-x}.$$

Let $f_1(x) = f(x)$ and for each $n = 2, 3, \ldots$, let $f_n(x) = f(f_{n-1}(x))$. What is the value of $f_{2012}(2012)$?

5. Evaluate $\int_0^{\frac{\pi}{2}} \ln(\sin x) dx$.

6. Let

$$I_\alpha = \int_0^\infty \frac{dx}{x^\alpha (1+x)}, \quad 0 < \alpha < 1.$$

Find the choice of α that minimizes I_α. Explain.

7. Let f be a continuous, decreasing function on $[0, 1]$. Show that

$$\int_0^1 f(x)(1-2x) dx \geq 0.$$

8. Evaluate

$$\int_0^\infty \frac{\arctan(\pi x) - \arctan(x)}{x} dx.$$

9. Let T be the triangle with vertices $(0, 0), (a, 0)$, and $(0, a)$. Find

$$\lim_{a \to \infty} a^4 e^{-a^3} \int_T e^{x^3+y^3} dx dy.$$