Concours Putnam

Atelier de Pratique
Le mardi, 26 septembre 12h30-13h30
5448 Pav. André Aisenstadt

1. Knowing that the expression

$$\sqrt{3+\sqrt{3+\sqrt{3+\dots}}}$$

converges, find its value.

> **Solution:** Let $\alpha = \sqrt{3+\sqrt{3+\sqrt{3+\dots}}}$. Then
>
> $$\alpha = \sqrt{3+\alpha}.$$
>
> This means that $\alpha^2 - \alpha - 3 = 0$. This means that $\alpha = \frac{1\pm\sqrt{13}}{2}$, since it has to be positive, we take the + sign.

2. Let $P(x)$ be a polynomial of degree n satisfying $P(k) = k$ for $k = 1, \dots, n$ and $P(0) = 1$. Find $P(-1)$.

> **Solution:** Let $Q(x) = P(x) - x$. Then $Q(x)$ is a polynomial of degree n whose roots are $k = 1, 2, \dots, n$. Thus $Q(x)$ is of the form $Q(x) = c\prod_{k=1}^{n}(x-k)$ for some constant c, and $P(x) = x + c\prod_{k=1}^{n}(x-k)$. Setting $x = 0$ gives $1 = P(0) = c(-1)^n n!$, so $c = \frac{(-1)^n}{n!}$. Hence,
>
> $$P(-1) = -1 + \frac{(-1)^n}{n!}\prod_{k=1}^{n}(-1-k) = -1 + (n+1) = n.$$

3. A car dealership that was open 7 days a week sold at least one car each day in 2015 and a total of 600 cars during that year. Prove that there was a period of consecutive days during which exactly 129 cars were sold. (There were 365 days in 2015).

> **Solution:** We present three solutions.
>
> *1st way.* Let a_n denote the number of cars sold in the first $1, 2, \dots, n$ days of the year. The assumption es that $1 \leq a_1 \leq a_2 \dots a_{365} = 600$. The a_n are distinct

positive integers in the range $\{1, 2, \ldots, 600\}$. Note that for $0 \leq i < j \leq 365$, $a_j - a_i$ is the number of cars sold on days $i+1, i+2, \ldots, j$. Thus, we need to show that $a_j = a_i + 129$ for some indices i and j. We apply the pigeon hole principle to the list of numbers $a_1, \ldots, a_{365}, a_1 + 129, \ldots, a_{365} + 129$. Since each a_i is an integer in the interval $[1, 600]$ these 730 integers are in the interval $[1, 729]$. By the pigeonhole principle, two fo these integers are the same. Since the a_i are all distinct and the same applies for the $a_i + 129$, this is only possible if $a_j = a_i + 129$ for some indices i and j.

2nd way. This solution gives the result under the weaker assumption that $a_{365} \leq 622$.

Let $a_0 = 0$, and consider the numbers $a_0, a_1, \ldots, a_{258}$. There are $2 \cdot 129 + 1$ of them, so the pigeonhole principle implies that there are at least three of them that are all congruent modulo 129, say $a_{i_1} \equiv a_{i_2} \equiv a_{i_3} \bmod 129$ with $0 \leq i_1 < i_2 < i_3 \leq 258$. We wish to show that either $a_{i_2} = a_{i_1} + 129$ or that $a_{i_3} = a_{i_2} + 129$. We argue by contradiction: if this were not the case, then we would have that $a_{i_2} - a_{i_1} \geq 2 \cdot 129$ and $a_{i_3} - a_{i_2} \geq 2 \cdot 129$ (because both of these differences are positive integers divisible by 129). But then

$$a_{258} = a_{258} - a_0 \geq (a_{i_3} - a_{i_2}) + (a_{i_2} - a_{i_1}) \geq 4 \cdot 129 = 516.$$

We conclude that

$$a_{365} = (a_{365} - a_{258}) + a_{258} \geq (365 - 258) + 516 = 623,$$

a contradiction.

3rd way. Similar to the second: shorter but less optimized.

Since $a_1 \geq 1$ and $a_{i+1} \geq a_i + 1$ for all $1 \leq i < 365$, we have $a_i \geq i$ for all $1 \leq i \leq 365$ and $a_{365-k} \leq 600 - k$ for all $0 \leq k < 365$. The latter inequality implies that $a_i + 129 \leq 600$ for $1 \leq i \leq 365 - 129 = 236$. Now if $a_j - a_i \neq 129$ for all $1 \leq i < j \leq 365$, there must be $365 + 236 = 601$ different integers in $[1, 600]$, which is a contradiction.

4. A car travels from one city to another at a rate of 40 miles per hour and then returns at a rate of 60 miles per hour. What is the average rate for the round trip?

Solution: Let D be the distance between the cities in miles. Then the total time for the trip is $T = \frac{D}{40} + \frac{D}{60}$ hours. The average rate of speed for the round trip is

$$\frac{2D}{\frac{D}{40} + \frac{D}{60}} = 48 \text{ mph}$$

5. Transportania is a country with finitely many cities, each of which is directly connected by a road with exactly three other cities. Thus, a traveler who arrives at a city along one of the three roads leading into it can choose between the two other roads, one to his left and one to his right, to continue his trip, assuming that he does not want to return to the city he just came from. Suppose that a traveler starts at city A, goes to city B, there takes the road to his right to city C, then takes the road to his left to city D, and so on, altenating between the left and the right road. Prove that he eventually gets back to city A.

Solution: Suppose each road is a 4-lane road with 2 lanes in each direction, and suppose the traveler takes the left lane if the road he has taken was his "left" choice, and the right lane if the road was his "right" choice. Given the rule of alternating left and right turns, a particular lane determines the itinerary completely forwards and backwards. Since there are finitely many cities, the number of roads and the number of lanes are also finite, and at some point the traveler must hit the same lane twice. Since this lane determines the complete itinerary in both directions, his entire itinerary must be a closed loop and in particular cannot contain any "feeders". Hence, he must revisit any place that he has visited before.

Another way.
Let V be the set of cities and E the set of roads. Then (V, E) forms a graph with every vertex of degree 3. Consider the set P of all tuples $\{(v, e, \epsilon) \mid v \in V\}$ where e is an edge adjacent to v, which we think of as the road along which we enter a city and $\epsilon \in \{\pm 1\}$ is the choice of direction in which we turn when exiting it: -1 corresponds to "left" and 1 corresponds to "right". Consider the map $T : P \to P$ given by $T(v, e, \epsilon) = (v', e', -\epsilon)$, where e' is the edge adjacent to v obtained from turning in direction ϵ when exiting city v if we entered by way of e and v' is the other endpoint of e'. This map is injective, since its inverse is $S(v', e', \epsilon') = (v, e, -\epsilon)$ where v is the other endpoint of e' and e is obtained by turning in direction $-\epsilon$ when exiting v if we entered by way of e'. Now since P is a finite set, so is the set of bijections from P to P, hence by the pigeonhole principle $T^k = T^l$ for some $k > l$. Since T is a bijection, $T^N = \text{id}$ for $N = k - l > 1$. Note that $Tx \neq x$ for all $x \in P$. In particular if we start with $x = (A, e_0, -1)$, $(T^m x)_{x \in \mathbb{N}}$ describes the itinerary of the car and $T^N x = x$ so we returned to x and to city A in particular.