**Concours Putnam**

Atelier de Pratique

Le lundi, 12 novembre 12h30-13h30 (Salle: Pavillon André-Aisenstadt 5448)

Polynômes

**Factor Theorem** The polynomial \( p(x) = a_n x^n + \ldots + a_1 x + a_0 \) has a root \( \alpha \) of multiplicity \( m \), then \( p(x) = (x-\alpha)^mq(x), q(\alpha) \neq 0 \).

**Elementary Symmetric Polynomials** Every symmetric polynomial in \( x_1, x_2, \ldots, x_n \) can be expressed as a polynomial in \( \sigma_1, \sigma_1, \ldots, \sigma_n \), where

\[
\sigma_k = \sum_{1 \leq j_1 < j_2 < \ldots < j_k \leq n} x_{j_1} x_{j_2} \ldots x_{j_k}
\]

**Vieta’s Formula** Let \( z_1, z_2, \ldots z_n \) be the (possibly complex) roots of the monic polynomial \( p(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 \). Then \( a_{n-k} = (-1)^k \sigma_k(z_1, z_2, \ldots, z_n) \) where \( \sigma_k \) is the elementary symmetric polynomial of degree \( k \) in \( n \) variables.

**Identity Theorem** If \( p(x) \) and \( q(x) \) are polynomials of degree at most \( n \), and \( p(x_k) = q(x_k) \) for \( 1 \leq k \leq n+1 \) for distinct \( x_1, x_2, \ldots, x_{n+1} \), then \( p(x) = q(x) \) for all \( x \).

1. Let \( \alpha = 2^{1/3} + 5^{1/2} \). Find a polynomial \( p(x) \) with integer coefficients satisfying \( p(\alpha) = 0 \).

**Solution:** \((\alpha - \sqrt{5})^3 = 2 \Rightarrow \alpha^3 + 15\alpha - 2 = \sqrt{5}(3\alpha^2 + 5)\). Therefore, \((\alpha^3 + 15\alpha - 2)^2 - 5(3\alpha^2 + 5)^2 = 0\). Thus,

\[
p(x) = (x^3 + 15x - 2)^2 - 5(3x^2 + 5)^2 = x^6 - 15x^4 - 4x^3 + 75x^2 - 60x - 121
\]

has the required property.

2. Find a polynomial of degree at most 3 such that \( p(2) = 3, p(3) = 5, p(5) = 8 \) and \( p(7) = 13 \).

**Solution:** Let \( p(x) = a + b(x-2) + c(x-2)(x-3) + d(x-2)(x-3)(x-5) \). Then

\[
a = p(2) = 3; \quad p(3) = a + b \Rightarrow b = 2; \quad p(5) = a + 3b + 6c \Rightarrow c = -1/6; \quad p(7) = a + 5b + 20c + 40d \Rightarrow d = -1/12.
\]

Thus,

\[
p(x) = 3 + 2(x-2) - \frac{(x-2)(x-3)}{6} - \frac{(x-2)(x-3)(x-5)}{12}.
\]

3. If \( x + y + z = 3, x^2 + y^2 + z^2 = 5, x^3 + y^3 + z^3 = 7 \), find \( x^4 + y^4 + z^4 \).
6. Suppose that the monic polynomial \( P(x, y, z) = x + y + z, \sigma_2(x, y, z) = xy + yz + zx \) and \( \sigma_3(x, y, z) = xyz \) denote the elementary symmetric polynomials in \( x, y, \) and \( z \). We have, \( \sigma_1 = 3, \sigma_1^2 - 2\sigma_2 = 5 \) and \( 7 - 3\sigma_3 = \sigma_1(\sigma_1^2 - 3\sigma_2) \). Thus, \( \sigma_2 = 2 \) and \( \sigma_3 = -2/3 \). Now \( x^4 + y^4 + z^4 = (x^2 + y^2 + z^2)^2 - 2(\sigma_2^2 - 2\sigma_1\sigma_3) = 9 \).

Solution: Let \( \sigma_1(x, y, z) = x + y + z, \sigma_2(x, y, z) = xy + yz + zx \) and \( \sigma_3(x, y, z) = xyz \) denote the elementary symmetric polynomials in \( x, y, \) and \( z \). We have, \( \sigma_1 = 3, \sigma_1^2 - 2\sigma_2 = 5 \) and \( 7 - 3\sigma_3 = \sigma_1(\sigma_1^2 - 3\sigma_2) \). Thus, \( \sigma_2 = 2 \) and \( \sigma_3 = -2/3 \). Now \( x^4 + y^4 + z^4 = (x^2 + y^2 + z^2)^2 - 2(\sigma_2^2 - 2\sigma_1\sigma_3) = 9 \).

4. Find all polynomials \( P(x) \) satisfying \( P(x^2 + 1) = (P(x))^2 + 1 \) for all \( x \) and \( P(0) = 0 \).

Solution: Consider the sequence \( \{u_k\} \) defined as follows: \( u_0 = 0; u_k = u_{k-1}^2 + 1 \) for \( k \geq 1 \). It can be easily proved by induction on \( k \) that \( P(u_k) = u_k \) for all \( k \). Since \( u_k > u_{k-1} \) for all \( k \), \( P(x) \) coincides with \( x \) for infinitely many values. It follows from the Identity Theorem that \( P(x) = x \).

5. Find a non-zero polynomial \( P(x, y) \) such that \( P([t], [2t]) = 0 \) for all real numbers \( t \).

Solution: This is (Putnam '05, B1). Answer: Let \( [t] = n \). Thus \( n \leq t < n + 1 \), i.e, \( 2n \leq 2t < 2n + 2 \). It follows that \( [2t] = 2[t] \) or \( [2t] = 2[t] + 1 \). Thus \( P(x, y) = (y - 2x)(y - 2x - 1) \) satisfies \( P([t], [2t]) = 0 \) for all \( t \).

6. Suppose that the monic polynomial \( p(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1x + 1 \) has non-negative coefficients and \( n \) real roots. Show that \( p(2) \geq 3^n \).

Solution: Let \( y_1, y_2, \ldots, y_n \) be the roots of \( p(x) \). Since \( p(x) \geq 1 \) when \( x \geq 0 \), we have \( y_i < 0 \) for all \( i \). Let \( y_i' = -y_i \). Note that

\[
p(2) = (2 + y_1')(2 + y_2') \ldots (2 + y_n').
\]

By Vieta’s formula, \( y_1'y_2' \ldots y_n' = 1 \). Also, by the AGM inequality, \( 2 + y_i' \geq 3(y_i')^{1/3} \). It follows that \( p(2) \geq 3^n \).

Another solution. Again notice that the roots \( y_i < 0 \). Thus we have,

\[
p(2) = \sum_{k=0}^{n} (-1)^k \sigma_k(y_1, \ldots, y_n) 2^{n-k} = \sum_{k=0}^{n} \sigma_k(|y_1|, \ldots, |y_n|) 2^{n-k}.
\]

By AGM,

\[
\frac{\sigma_k(|y_1|, \ldots, |y_n|)}{\binom{n}{k}} \geq \left( \frac{n}{k} \right)^{k} \frac{1}{\sqrt[k]{y_1 \ldots y_n}} = 1.
\]

Therefore,

\[
p(2) \geq \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} = (1 + 2)^n = 3^n.
\]
7. Let \( p(x) = a_n x^n + \ldots + a_1 x + a_0 \) be a polynomial with integer coefficients. If \( r \) is a rational root of \( p(x) \), show that the numbers \( a_n r, a_n r^2 + a_{n-1} r, \ldots, a_n r^n + a_{n-1} r^{n-1} + \ldots + a_1 r \) are all integers.

**Solution:** This (Putnam ’04, B1). Let \( r = b/c \), with \((b, c) = 1\) (i.e., \( b \) and \( c \) are relatively prime). Since \( p(r) = 0 \), we get, after clearing denominators,

\[
a_n b^n + a_{n-1} b^{n-1} c + \ldots + a_0 c^n = 0.
\]

For \( 1 \leq k \leq n \), define \( p_k(b, c) = a_n b^n + a_{n-1} b^{n-1} c + \ldots + a_{n-k+1} b^{n-k+1} c^{k-1} \).

Note that \( c^k | p_k(b, c) \). But \( p_k(b, c) = b^{n-k}(a_n b^k + a_{n-1} b^{k-1} + \ldots + a_{n-k+1} b c^{k-1}) \). Furthermore, \((b, c) = 1 \Rightarrow (b^{n-k}, c^k) = 1\). Thus,

\[
c^k | a_n b^k + a_{n-1} b^{k-1} c + \ldots + a_{n-k+1} b c^{k-1}.
\]

It follows that \( a_n r^k + a_{n-1} r^{k-1} + \ldots + a_{n-k+1} r \) is an integer for \( 1 \leq k \leq n \).

8. Do there exist polynomials \( a(x), b(x), c(y), d(y) \) such that \( 1 + xy + x^2 y^2 = a(x)c(y) + b(x)d(y) \)?

**Solution:** This is (Putnam ’03, B1)

Suppose that such polynomials exist. Then \( 1 = c(0)a(x) + d(0)b(x), 1 + x + x^2 = c(1)a(x) + d(1)b(x), 1 - x + x^2 = c(-1)a(x) + d(-1)b(x) \). We have

\[
1 = c(0)a(x) + d(0)b(x),
\]
\[
x = \frac{c(1) - c(-1)}{2} a(x) + \frac{d(1) - d(-1)}{2} b(x),
\]
\[
x^2 = \frac{c(1) + c(-1) - 2c(0)}{2} a(x) + \frac{d(1) + d(-1) - 2d(0)}{2} b(x).
\]

So we have three linearly independent elements (the polynomials \( 1, x, x^2 \)) in a subspace of dimension 2 (the vector space spanned by \( a(x) \) and \( b(x) \)). Contradiction.