Concours Putnam
Atelier de Pratique
Le lundi, 12 novembre 12h30-13h30 (Salle: Pavillon André-Aisenstadt 5448)
Polynômes

Factor Theorem The polynomial \( p(x) = a_n x^n + \ldots + a_1 x + a_0 \) has a root \( \alpha \) of multiplicity \( m \), then \( p(x) = (x - \alpha)^m q(x), q(\alpha) \neq 0 \).

Elementary Symmetric Polynomials Every symmetric polynomial in \( x_1, x_2, \ldots, x_n \) can be expressed as a polynomial in \( \sigma_1, \sigma_1, \ldots, \sigma_n \), where
\[
\sigma_k = \sum_{1 \leq j_1 < j_2 < \ldots < j_k \leq n} x_{j_1} x_{j_2} \ldots x_{j_k}
\]

Vieta’s Formula Let \( z_1, z_2, \ldots z_n \) be the (possibly complex) roots of the monic polynomial \( p(x) = x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \). Then \( a_{n-k} = (-1)^k \sigma_k (z_1, z_2, \ldots, z_n) \) where \( \sigma_k \) is the elementary symmetric polynomial of degree \( k \) in \( n \) variables.

Identity Theorem If \( p(x) \) and \( q(x) \) are polynomials of degree at most \( n \), and \( p(x_k) = q(x_k) \) for \( 1 \leq k \leq n + 1 \) for distinct \( x_1, x_2, \ldots, x_{n+1} \), then \( p(x) = q(x) \) for all \( x \).

1. Let \( \alpha = 2^{1/3} + 5^{1/2} \). Find a polynomial \( p(x) \) with integer coefficients satisfying \( p(\alpha) = 0 \).
2. Find a polynomial of degree at most 3 such that \( p(2) = 3, p(3) = 5, p(5) = 8 \) and \( p(7) = 13 \).
3. If \( x + y + z = 3, x^2 + y^2 + z^2 = 5, x^3 + y^3 + z^3 = 7 \), find \( x^4 + y^4 + z^4 \).
4. Find all polynomials \( P(x) \) satisfying \( P(x^2 + 1) = (P(x))^2 + 1 \) for all \( x \) and \( P(0) = 0 \).
5. Find a non-zero polynomial \( P(x, y) \) such that \( P([t], [2t]) = 0 \) for all real numbers \( t \).
6. Suppose that the monic polynomial \( p(x) = x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + 1 \) has non-negative coefficients and \( n \) real roots. Show that \( p(2) \geq 3^n \).
7. Let \( p(x) = a_n x^n + \ldots + a_1 x + a_0 \) be a polynomial with integer coefficients. If \( r \) is a rational root of \( p(x) \), show that the numbers \( a_n r, a_n r^2 + a_{n-1} r, \ldots, a_n r^n + a_{n-1} r^{n-1} + \ldots + a_1 r \) are all integers.
8. Do there exist polynomials \( a(x), b(x), c(y), d(y) \) such that \( 1 + xy + x^2 y^2 = a(x)c(y) + b(x)d(y) \)?